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NEW SERIES

OF THE

MATHEMATICAL

REPOSITORY.

By THOMAS LEYBOURN,

Of the Royal Military College.

VOL. I.

London:

PRINTED AND SOLD BY W. GLENDINNING, NO. 25,
HATTON GARDEN.

1806

ADVERTISEMENT.

The first volume of the New Series of the Mathematical Repository being now compleated, it may be proper to state briefly its plan. The volume consists of three distinct parts. The first contains one hundred and twenty Questions, in almost every branch of the Mathematics. These are, with a few exceptions entirely new, and each Question has at least one, but in general two or more answers, furnished by different ingenious Mathematicians.

The utility of this part of the work will be readily admitted when it is considered, that almost all the improvements which the Mathematics have received, have originated in the exertions made to resolve particular problems, such as that of the *trisection of an angle, the duplication of the cube*, &c. among the ancients; also the various *isoperimetrical problems*, and above all, the problem of *the three bodies* among the moderns. We believe also, that most Mathematicians will confess how much their talents have been cultivated and their knowledge improved, by resolving problems, such as are proposed in this volume.

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The first number of the second volume is now published; the remaining three numbers shall follow, one every six months, so that a second volume will be compleated in the course of eighteen months from this date.

R. M. College, 1st. December, 1806.

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SECOND SOLUTION, *by Mr.* WILLIAM MARRAT, *Boſton.*

Fig. 4. Pl. 1. Let ABC repreſent the earth, D its centre, DC the radius drawn perpendicular to the tangent EC, CFG a chord perpendicular to DE, and BE the height required; alſo ſuppoſe the perſon's weight at B to be = unity, the radius DB = DC = 3979 miles = a, and $p = 3{\cdot}14159$, &c. Then, the force of gravity above the ſurface being inverſely as the ſquare of the diſtance from the centre, it will be as $\frac{4}{5} : 1 :: a^2 : \frac{5a^2}{4}$ = the ſquare of the diſtance from the centre of the earth; therefore $\sqrt{\frac{5a^2}{4}} = \frac{a}{2}\sqrt{5}$ = DE, the diſtance itſelf, and DE $-$ DB $= \frac{a}{2}\sqrt{5} - a = 469{\cdot}656$ = BE, the height above the ſurface. Now, by ſimilar triangles, DE : DC :: DC : DF, *i. e.* $\frac{a}{2}\sqrt{5} : a :: a : \frac{2}{5}a\sqrt{5}$, and $a - \frac{2}{5}a\sqrt{5} = a \times (1 - \frac{2}{5}\sqrt{5})$ is the height of the ſegment; therefore, (by Prob. 10, pa. 99, of Dr. Hutton's Menſuration, Edit. 3.)

$$2ap \times a \times \left(1 - \frac{2}{5}\sqrt{5}\right) = 2a^2 p \times \left(1 - \frac{2}{5}\sqrt{5}\right)$$

$= 10502188{\cdot}84$ ſquare miles, is the portion of the ſurface that could be ſeen by a ſpectator at E.

THIRD SOLUTION, *by* MERONES MINOR.

The force of gravity being inverſely as the ſquare of the diſtance from the earth's centre, putting $r = 3979$, $p = 3{\cdot}14159$, &c. and x = the diſtance required, we have $\frac{1}{r^2} : \frac{1}{x^2} :: 1$, the force of gravity at the diſtance $r : \frac{4}{5}$; hence, $x^2 : r^2 :: 5 : 4$, or $x : r :: \sqrt{5} : 2$, and $x = \frac{r\sqrt{5}}{2}$. Then, by Menſuration, the area of the viſible part of the ſurface is 10502188 84 ſquare miles.

Other ingenious Solutions were given by Meſſrs. Boole, Cavill, *C[illegible]iling, Collins,* Hearding, Hopper, Morley, *and* Thompſon,

V. QUESTION 5, *by Mr.* JOHN LOWE, *Attercliffe.*

To find three numbers in arithmetical progreſſion ſuch that the ſum of every two of them diminiſhed by the third ſhall be a ſquare number.

FIRST SOLUTION, *by Mr.* J. I.

Let a, b, c be the three numbers required, then by the conditions of the queſtion

$$a + c = 2b$$
$$a + c - b = x^2$$
$$a + b - c = y^2$$
$$c + b - a = z^2.$$

Let b be exterminated from the two laſt equations by means of the firſt, and there will reſult

$$\frac{3}{2}a - \tfrac{1}{2}c = y^2$$

$$\frac{3}{2}c - \tfrac{1}{2}a = z^2$$

whence $a = \frac{1}{4}(3y^2 + z^2)$ and $c = \frac{1}{4}(3z^2 + y^2)$.

Combining the two firſt equations it will eaſily be found that $b = x^2$: and putting the values of a, b, and c that have been found in the firſt equation, we get

$$y^2 + z^2 = 2x^2$$

$$\text{or } \left(\frac{y+z}{2}\right)^2 + \left(\frac{y-z}{2}\right)^2 = x^2.$$

And this laſt equation is the only condition neceſſary; the numbers x, y, and z being in every other reſpect arbitrary or indeterminate. We may, therefore, aſſume for $\frac{y+z}{2}$, $\frac{y-z}{2}$ and x, any three numbers that meaſure the ſides of a rational right-angled triangle, and hence innumerable ſolutions may be derived.

The leaſt numbers that form a right-angled triangle are 4, 3, and 5: put $\frac{y+z}{2} = 4$, $\frac{y-z}{2} = 3$, and $x = 5$; then $y = 7$ and $z = 1$: whence

$b =$

$$b = x^2 = 25$$
$$a = \tfrac{1}{4}(3y^2 + z^2) = 37$$
$$c = \tfrac{1}{4}(3z^2 + y^2) = 13$$

And theſe are the leaſt numbers that ſatisfy the queſtion.

SECOND SOLUTION, *by Mr.* JOHN WRIGHT, *Norley.*

Let x, y and z denote the numbers ſought. Then, ſince $x + y - z$, $x + z - y$, and $y + z - x$ is each to be a ſquare number, let them be equated with a^2, b^2, and c^2 reſpectively, and we ſhall thence obtain

$$x = \frac{a^2 + b^2}{2},\ y = \frac{a^2 + c^2}{2},\ \text{and } z = \frac{b^2 + c^2}{2}.$$

Now per queſtion $x + z = 2y$, that is, $\frac{a^2 + 2b^2 + c^2}{2} = a^2 + c^2$, or $a^2 + c^2 = 2b^2$: hence muſt $a = 2rs + r^2 - s^2$, $b = r^2 + s^2$, and $c = 2rs + s^2 - r^2$, as is well known. Theſe expreſſions for a, b, and c being written inſtead of them in the equations x, y, and z above, they become

$x = (r^2 + s^2)^2 + 2rs(r^2 - s^2)$, $y = r^2 + s^2$ and $z = (r^2 + s^2)^2 + 2rs(s^2 - r^2)$, where r and s may be expounded by any two numbers taken at pleaſure.

Ex. Gr. Take $r = 1$, and $s = 2$; then $x = 13$, $y = 25$, and $z = 37$.

THIRD SOLUTION, *by Mr.* THOMAS BAZLEY, *Bolton.*

Let $x^2 - y$, x^2 and $x^2 + y$ be the three numbers; then x^2, $x^2 - 2y$, and $x^2 + 2y$ are to be ſquares; the firſt is evidently ſo, and for the ſecond aſſume

$x^2 - 2y = x^2 - 2(2rx - 2r^2) = x^2 - 4rx + 4r^2 = (x - 2r)^2$, a ſquare.

Now for the third $x^2 + 2y$, or $x^2 + 4rx - 4r^2$, aſſume it $= z^2$;

then $z^2 - x^2 = 4rx - 4r^2$, or $(z + x) \times (z - x) = 4r(x - r)$; therefore, by Mr. Leſlie's Principle *.

$(z + x) \times m(z - x) = 4rm(x - r)$, or $z + x = 4rm$, and $m(z - x) = x - r$

whence $z = 4rm - x = \frac{x - r + mx}{m}$; hence $x = \frac{4rm^2 + r}{2m + 1}$;

---

* *See Mathematical* Repository, Vol. I. Old Series.

and, conſequently, $y = 2rx - 2r^2 = \frac{8r^2m^2 + 2r^2}{2m+1} - 2r^2$, which is eaſily reduced to $\frac{4r^2(2m^2 - m)}{2m+1}$; and, therefore, the three numbers are,

$$\left(\frac{4rm^2+r}{2m+1}\right)^2 - \frac{4r^2(2m^2-m)}{2m+1};\ \left(\frac{4rm^2+r}{2m+1}\right)^2 \& \left(\frac{4rm^2+r}{2m+1}\right)^2 + \frac{4r^2(2m^2-m)}{2m+1}:$$

or to have the reſult in whole numbers, we ſhall get, by taking away the denominators,

$$(4rm^2+r)^2-4r^2m(4m^2-1);(4rm^2+1)^2\&(4rm^2+r)^2+4r^2m(4m^2-1).$$

Now, if r and m be taken each equal to unity, we have 13, 25, and 37 for the three leaſt whole numbers that anſwer the conditions of the queſtion.

FOURTH SOLUTION, *by Mr.* MARRAT, *Boſton.*

Let $x^2 - y$, x^2, and $x^2 + y$ denote the progreſſion; then, by the queſtion $x^2 - 2y$, $x^2 + 2y$, and x^2 are to be ſquare numbers: the laſt is a ſquare.

Now, ſuppoſe $x^2 - 2y = (q - x)^2$, and $x^2 + 2y = (p - x)^2$; then, by the firſt $y = qx - \frac{1}{2}q^2$, and from the ſecond $y = \frac{1}{2}p^2 - px$; therefore, $qx - \frac{1}{2}q^2 = \frac{1}{2}p^2 - px$, and $x = (p^2 + q^2) \div (2p + 2q)$, where p and q may be taken at pleaſure. If $p = 2$, and $q = 1$, then $x = \frac{5}{6}$ and $y = \frac{1}{2}p^2 - px = \frac{2}{6}$, therefore, $x^2 - y = \frac{13}{30}$, $x^2 = \frac{25}{26}$ and $x^2 + y = \frac{37}{36}$; and, therefore, rejecting the common denominators, the numbers are 13, 25, and 37.

FIFTH SOLUTION, *by Mr.* JOHN CORNTHWAITE.

Put $x^2 + x$, $x^2 + 2x$ and $x^2 + 3x$ for the three numbers; then the difference between the ſum of the firſt and ſecond and the third is equal to x^2. Hence we have only to make $x^2 + 2x$ and $x^2 + 4x$ ſquares; the difference of theſe is $2x$, which being reſolved into the factors $2x$ and 1, we may put $(x + \frac{1}{2})^2$, the ſquare of half the ſum of the factors equal to the greater of the expreſſions to be made ſquares; that is, we may put $(x + \frac{1}{2})^2 = x^2 + 4x$: *this equation reduced* gives $x = \frac{1}{12}$; and the three numbers are $\frac{13}{144}$, $\frac{25}{144}$, and $\frac{37}{144}$; or, rejecting the denominators 13, 25, and 37.

Other Solutions were received from Meſſrs. Boole, Cavill, Codling, Collins, Lowe, Morley, Myers, Pickering, Scurr, *and* Thompſon.

VI. QUESTION 6, *by Mr.* T. MILNER, *Sheffield.*

If two rings or circles of equal radii touch each other on their circumferences at P, and one of them be made to revolve round the periphery of the other, ſo that the area deſcribed by the point P may be equal to 2·68 ſquare inches. *Query* the diameter of a circle equal to that of either ring?

FIRST SOLUTION, *by* SCOTICUS.

Let A (Fig. 5, Pl. 1.) be the centre of the fixed circle, in which P is the point of contact of the two circles at the beginning of the motion; let B be the point of contact when the revolving circle BP′E has any other poſition, take the arch BP′ in the revolving circle equal to the arch BP of the fixed circle, then P′ will evidently be a point in the curve, which is the *locus* of the revolving point. Join PP′ (which will be parallel to ABC a line joining the centres of the circles) and draw PD, P′D′ perpendicular to AC. Put r for the radius of the equal circles, and φ for the indefinite angle PAB or FPP′ then $BD = BD' = r\,(1 - \text{coſ.}\,\varphi)$ and $PP' = DD' = 2r\,(1 - \text{coſ.}\,\varphi)$, and ſince the fluxion of the area deſcribed by the line PP′ revolving about P is equal to ½PP′ multiplied into the fluxion of the circular arch whoſe radius is PP′, that is to

$$2r^2\dot{\varphi}\,(1 - \text{coſ.}\,\varphi)^2 = r^2\dot{\varphi}\,(3 - 4\,\text{coſ.}\,\varphi + \text{coſ.}\,2\varphi);$$

by taking the fluent, we have the area itſelf equal to

$$r^2\,(3\varphi - 4\,\text{ſin.}\,\varphi + \tfrac{1}{2}\,\text{ſin.}\,2\varphi):$$

which expreſſion, by taking $\varphi = 360°$ gives the whole area $= 3 \times 2 \times 3{\cdot}1416 \times r^2$; but, by the queſtion, the area $= 2{\cdot}68$ ſquare inches: hence $2r$, the diameter of either circle is eaſily *found* $= {\cdot}7541$ the anſwer.

SECOND SOLUTION, *by Mr.* JOHN WRIGHT, *Norley.*

Let O (Fig. 6, Pl. 1.) be the centre, and PF a diameter of the circle or ring PIF, about which the equal circle AIH is ſuppoſed to roll; with the centre F and radius FP deſcribe a circle cutting PF produced in C, and the circle AIH in H: then, if the chord HA be a tangent to the circle PHC at H, the point A will be the ſituation of P, when the circles are in contact at I. For, draw the radius OI, which produce till it meet the chord AH in D, and join AI, FH, PI, and HI. Then, ſince the circles touch each other at I, it is manifeſt that AD = DH, and becauſe PO = FO, the lines PA, OD, FH are parallel: conſequently, the points P, I, H, (PO, PF being reſpectively equal to OI and FH) are in a ſtraight line: hence PI = IH = AI, and the arch PI = AI, or A is a point in the curve PAC deſcribed by the motion of the point P.

Moreover, demit HQ perpendicular to PF and parallel thereto draw the radius FN cutting HB parallel to PF in B. Since FH is parallel to PA, it is well known that the fluxion of the curvilinear ſegment PAP is

$$= \frac{PA^2 \cdot \dot{P}H}{2FH} = \frac{PQ^2 \cdot \dot{P}Q}{2HQ} = \frac{(PF \pm \frac{1}{2}FQ) \cdot QF \cdot Q\dot{F}}{HQ}$$

$$\mp \frac{\frac{1}{2}PF^2 \cdot \dot{Q}F}{HQ} = - PF \cdot H\dot{Q} \pm \tfrac{1}{2}FQ \cdot H\dot{Q} \mp \tfrac{1}{2}PF \cdot H\dot{N}$$

$= - FN \cdot H\dot{Q} \mp \frac{1}{2}HB \cdot B\dot{N} \mp \frac{1}{2}PF \cdot H\dot{N}$; the fluent or quadrature PAP is, therefore, FN·HQ $\mp$ ½ ſeg. HBN $\mp$ ſector FHN, which being corrected by ſubtracting therefrom its value at the initial point of the curve, we ſhall have — FN·HQ $\mp$ ½ ſeg. HBN ± ſector FHN + $\frac{3}{2}$ quadrant PFN, for the true quadrature PAP; and this, when H coincides with C, becomes $\frac{3}{2}$ of the ſemi-circle PNC. Hence the area of the whole curve PACP, deſcribed by a complete revolution of P, is equal to 3 of the ſemi-circles PNC, and, conſequently, the ſemi-circle PNC = ⅓ of the given quadrature; that is $= \frac{2 \cdot 68}{3}$, and its radius or the required diameter of the ring FP = ·7541.

Other Solutions were given by Meſſrs. Boole, Cavill, Codling, Collins, Milner, Morley, Myers, Pickering, Scurr, and *Thompſon.*

VII. QUESTION 7, *by Mr.* JOHN WHITLEY, *Attercliffe.*

Let FE, the diameter of a circle deſcribed about any plane triangle ACB, biſect the baſe AB perpendicularly in H, and let EP, FS be drawn perpendicular to AC: with the radii CH, CS and centre C deſcribe the circles CQG and CSD meeting AC in G and S; and let S be the neareſt point to C, then from P and G draw the tangents PQ, GD and the rectangle PQ, GD will be equal to the area of the triangle ACB. Required the demonſtration.

FIRST SOLUTION, *by Mr.* THOMAS BAZLEY, *Bolton.*

Fig. 7, Pl. 1, being conſtructed as per queſtion, it is well known that CS = AP = half the difference, and CP or AS = half the ſum of the ſides of the triangle.

Now, by well known properties of triangles, we have

$CH^2 = CQ^2 = \frac{1}{2}(AC^2 + CB^2 - \frac{1}{2}AB^2)$, alſo $CP^2 = (\frac{AC+CB}{2})^2$; whence $CP^2 - CQ^2 = PQ^2 = \left(\frac{AB}{2}\right)^2 - \left(\frac{AC-CB}{2}\right)^2$, alſo $GD^2 = CG^2 - CD^2 = CH^2 - CS^2 = \frac{1}{2}(AC^2 + BC^2 - \frac{1}{2}AB^2) - (\frac{AC-CB}{2})^2$; that is, $GD^2 = (\frac{AC+CB}{2})^2 - (\frac{AB}{2})^2$ as will eaſily appear by expanding the expreſſions. Hence we obtain PQ × GD a mean proportional between theſe two quantities $(\frac{AB}{2})^2 - (\frac{AC-CB}{2})^2$ and $(\frac{AC+CB}{2})^2 - (\frac{AB}{2})^2$; which is ſhewn to be equal to the area of the triangle at Cor. 2. Prop. 3. B. II. of Emerſon's Geometry.

SECOND SOLUTION, *by Mr.* THOS. HOPPER, *Mancheſter.*

Fig. 8, Pl. 1. The figure being conſtructed as directed by the queſtion, join CE, and let OM be the radius of the inſcribed circle, and PK the radius of a circle touching the baſe, and the continuation of the ſides, then PK = PM = AH; AP = SC = half the difference of the ſides, CP = half the ſum of *the ſides, and* CK = half the perimeter. From P through H draw *PL meeting CP in L;*

Then $HC^2 - HE^2 = PC^2 - PE^2$ (Simp. Geo. 9.2.); conſequently, $PC^2 - HC^2 = PE^2 - HE^2$:
But $PQ^2 = PC^2 - PH^2$; alſo if CR be drawn parallel to AB and PR joined, we ſhall have $PE^2 = REH = RHE + HE^2$; conſequently, $PE^2 - HE^2 = RHE$;
and, therefore, $PQ^2 = RHE = ANB = IK \times ON$, a well-known property.

Alſo $2AH^2 + 2CH^2 = AC^2 + CB^2$ (Simp. Geo. 11. 2.) $= 2PC^2 + 2AP^2$ ($2CS^2$);
conſequently, $CH^2 - CS^2 = PC^2 - AH^2$: but $GD^2 = CH^2 - CS^2$; therefore, $GD^2 = PC^2 - AH^2 = (PC + AH)(PC - AH) = (PC + PK)(PC - PM) = CK \cdot CM$.

Now, ſince $PQ^2 = IK \times ON$ and $GD^2 = CK \cdot CM$, it will be as $IK : PQ :: PQ : OM$, and $CM : GD :: GD : CK$;
therefore, $IK \times CM : PQ \times GD :: PQ \times GD : OM \times CK$:

But it is well known that $IK \times CM = OM \times CK =$ the area of the triangle ABC. Conſequently, $PQ \times GD$ will alſo be equal to the area of the triangle ABC.

Q. E. D.

THIRD SOLUTION, *by Mr.* JOHN BUTTERWORTH, *Haggate, near Oldham.*

The figure being drawn as before. Then $PC^2 - CQ^2$ (CH^2) $= PQ^2$; but, by the theorem on page 39 of the Gentleman's Diary for 1803, $PC^2 - CH^2 = EH \cdot HR$, theref. $PQ^2 = EH \cdot HR$.

Again $CG^2 = (CH^2) - CD^2$ (CS^2) $= GD^2$, but $CH^2 = CR^2 + RH^2 = FR \cdot RE + RH^2$,
and $CS^2 = FR \cdot EH$; theref. $GD^2 = FR \cdot RE + RH^2 - FR \cdot EH = FR \cdot RH + RH^2 = FH \cdot RH$; and, conſequently,
$PQ^2 \cdot QD^2 = HE \cdot HR \cdot FH \cdot RH = AH^2 \cdot HK^2$, or $PQ \cdot GD = AH \cdot HK$.

Q. E. D.

FOURTH SOLUTION, *by Mr.* JOHN FAREY, *No.* 12, *Crown Street, Weſtminſter.*

Fig. 7, Pl. 1. Having conſtructed the figure as directed in the Queſtion, and alſo drawn the radii CQ and CD to the points of contact of the tangents PQ and GD; alſo joined CH, and drawn the perpendicular CI upon the baſe AB of the triangle; then, *becauſe of the right*-angled triangles PQC and GDC, we have $PC^2 - CQ^2 = PQ^2$, and $CG^2 - CD^2 = GD^2$; and ſince by the queſtion $PQ \times GD = \frac{1}{2}AB \times CI$, or $PQ^2 \times GD^2 = \frac{1}{4}AB^2 \times CI^2$, therefore, $(PC^2 - CQ^2) \times (CG^2 - CD^2) = \frac{1}{4}AB^2 \times CI^2$. Now, *(by Leybourn's Math. Repoſitory, Vol.* I. *p.* 284, *and Vol.* II. *p.* 101.*)* PC is equal to half the ſum of the ſides AC and AB of the triangle ACB, and CS or CD is equal to half the difference of the ſides of the ſame triangle; it being alſo evident from the conſtruction that CQ and CG are each equal to the line from the vertex biſecting the baſe of the ſame triangle, or CH. We may, therefore, infer from the queſtion, the following

THEOREM. *In any plane triangle, the difference between the ſquare of half the ſum of the ſides and the ſquare of the line biſecting the baſe, multiplied into the difference between the ſquare of the line biſecting the baſe and the ſquare of half the difference of the ſides, is equal to the ſquare of the area of the triangle.*

Before proceeding to demonſtrate this Theorem, I beg permiſſion to mention that it has long appeared to me a deſirable thing, to have the ſeveral properties of Triangles which the different geometrical and trigonometrical writers have demonſtrated, expreſſed each by an equation in the characters or ſymbols adopted by *Lawſon* and *Leybourn*, for expreſſing the parts of a triangle, in their *Synopſis* or *Tables of the Data*, from which triangles have been conſtructed; and I have been from time to collecting materials, which I hope that leiſure will permit me e'er long to arrange for publication, not only on the properties of triangles which can be thus expreſſed in an equation, but on the *loci* or *known parts* of a triangle, when one, two or three things relating thereto are *given*. As a ſpecimen of the facility with which certain known properties of a triangle when thus expreſſed, can be combined, to demonſtrate the above, and moſt other theorems, or to diſcover new properties; in the above Theorem I ſubſtitute (as is done in *Leybourn's Synopſis of Data for the Conſtructions of Triangles, page* iv.) B for the baſe, P for the perpendicular from the vertical angle (V) upon the baſe, S and s for the greater and leſs ſides (ſubtending V) reſpectively, m and n for the greater and leſs ſegments of the baſe by the perpendicular from V upon B, and l for the line from the vertex to the middle of the baſe, or line biſecting the baſe; and we have $\left(\left(\frac{S+s}{2}\right)^2 - l^2\right) \times \left(l^2 - \left(\frac{S-s}{2}\right)^2\right) = \frac{1}{4}B^2 \times P^2$. Or, by reducing the fractions *and arranging* the terms, to the order of their component parts

B, P, S, s, m, n, and l, as they ſhould be for more perfect arrangement in a general *Synoptical Table* of ſuch equations or properties, we have $4B^2 \times P^2 = ((S+s)^2-4l^2) \times (-(S-s)^2+4l^2)$; and by multiplying the terms of the laſt part of this equation, we get
$4B^2 \times P^2 = -S^4 + 2S^2 \times s^2 - s^4 + 8S^2 \times l^2 + 8s^2 \times l^2 - 16l^4 = -(S^2-s^2)^2 + 8l^2 \times (S^2+s^2-2l^2)$. Now, by *Leybourn's Repoſitory, Vol.* I. *p.* 17, *cor.* $\frac{1}{2}B^2 = S^2 + s^2 - 2l^2$; therefore, $4B^2 \times P^2 = -(S^2-s^2)^2 + 8l^2 \times \frac{1}{2}B^2$, and $4B^2 \times l^2 - 4B^2P^2 = (S^2-s^2)^2$, or $4B^2 \times (l^2 - P^2) = (S^2-s^2)^2$. But ſince $HC^2 - CI^2 = HI^2$ (ſee the figure) and $\frac{AI-IB}{2} = HI = \frac{m-n}{2}$, therefore, $l^2 - P^2 = (\frac{m-n}{2})^2$, and we have, $4B^2 \times \frac{1}{4}(m-n)^2 = (S^2-s^2)^2$; and, laſtly, by extracting the root, $B \times (m-n) = S^2 - s^2$, which is demonſtrated to be true by *Thomas Simpſon (ſee his Geometry Book* II. *prop.* 9, *cor.* 2.*)* and ſeveral other writers; our theorem, therefore, and the propoſition in the queſtion from whence it is derived are true.

Q. E. D.

Demonſtrations were alſo received from Meſſrs. Codling, Collins, Scoticus, *and* Whitley.

VIII. QUESTION 8, *by Mr.* I. T. M'DONELD.

If on any given right line AB, there be taken any variable diſtance AL, and from L, in the ſame direction, any given invariable diſtance LM; and if with the centres L, B and radii LA, BM arcs be deſcribed, it is required to determine the nature of the curve, which is the locus of P the point of their interſection?

SOLUTION *by Mr.* J. I.

Fig. 9, Pl. 1. Let $AB = a$; $LM = b$; $BM = BP = \rho$; and having drawn PO perpendicular to AB, put $BO = x$. Then $BL = \rho + b$; $LO = \rho + b - x$; $LP = AL = a - b - \rho$; Then, becauſe $LP^2 - LO^2 = BP^2 - BO^2$, we have in ſymbols $(a-b-\rho)^2 - (\rho + b - x)^2 = \rho^2 - x^2$; whence, $a^2 - 2(a-x)b = \rho^2 + 2(a-x)\rho$; and adding $-b^2 + b^2 + (a-x)^2$ to one ſide and its equal $(a-x)^2$ to the other ſide, there reſults $a^2-b^2+(a-b-x)^2=(a-x+\rho)^2$. Take AC = LM = b, draw CD perpendicular to AB, and make AD = AB = a; then $DC^2=a^2-b^2$; $CO^2=(a-b-x)^2$ and $(OA+BP)^2=(a-x+\rho)^2$; whence, we have DO = AO + PB or BP = DO — AO.

From what is done it will be eaſy to derive an algebraic equation for the rectangular co-ordinates of the curve; for we have only to put PO = y, to ſubſtitute $\surd(x^2+y^2)$ for ρ, and to clear the equation of radicals. The equation thus found will ſhew the curve to be of the fourth order; but the figure of the curve and its principal properties may be more readily deduced from the property inveſtigated above, viz. PB = DO — AO. The curve will conſiſt of two equal and ſimilar parts lying on different ſides of AB; it will be a ſort of oval incloſing the point B on every ſide. And if we take CH = CA and produce AB to K making 3BK = BH, the curve will cut the line AB in H and K.

Solutions were alſo given by Meſſrs. Bazley *and* Cavill.

IX. QUESTION 9, *by Mr.* JOHN WRIGHT, *Norley*.

From two given points in the diameter of a given ſemicircle, it is required, geometrically, to draw two right lines to meet each other at ſome point in the periphery, ſo that the ſum or difference of the reſpective rectangles under theſe two lines and two given ones (M and N) may be equal to a given magnitude.

FIRST SOLUTION, *by Mr.* WRIGHT, *the Propoſer*.

CONSTRUCTION. Let A and C (fig. 10, pl. 1.) be the two given points in the diameter DH of the given ſemicircle DBH, of which the centre is O. Produce OD to L till $OL \cdot OA = OD^2$, and make AN perpendicular to LC and equal to a mean proportional between LA and AC; alſo from A along AC or AL, according as the ſum or difference is given, ſet off, AE ſuch, that $AE^2 : AN^2 :: M^2 : P \times OC$ (P being a third proportional to AO and N); then through the points N, E, C deſcribe a circle, in which apply the chord NI, *ſuch,* that NI × N may be equal to the given ſum or difference*; and having,* on NI dropt the perpendicular CF, from C to the ſemicircle DBH apply $CB = FN$, then join AB and the thing is done.

DEMONSTRATION. Join OB, and parallel thereto draw CS meeting AB produced in S; alſo through the points L, A, B deſcribe a circle cutting CB produced in G, and join AG, CI, NE. Then, ſince the angles NEA, NIC are equal, the right-angled triangles NEA, FIC are ſimilar, and $IF^2 : FC^2 :: AE^2 : AN^2 :: M^2 : P \times OC$, by conſtruction, that is, $IF^2 : M^2 :: FC^2 : P \times OC$. Becauſe, by conſtruction, $AL \times AO = OD^2$ or OB^2, OB is a tangent to the circle LABG at B, and, therefore, the angle. $AGB = ABO = ASC$; hence the points A, G, S, C are in a circle, and $AB \cdot BS = GB \cdot BC = GC \cdot BC - BC^2$; but by the circle $GC \cdot BC = CL \cdot CA = CN^2$; wherefore, $AB \times BS = CN^2 - CB^2 (FN^2) = FC^2$. Now $AB^2 : AB \cdot BS (FC^2) :: AB : BS ::$ (by parallels) $AO : OC :: AO \times P : OC \times P$, and by alternation, $AB^2 : AO \times P = N^2 :: FC^2 : OC \times P$; hence $AB^2 : N^2 :: IF^2 : M^2$, or $AB : N :: FI (= IN \mp FN) : M$; therefore, $AB \times M = IN \times N \mp FN \times N (= BC \times N)$, and conſequently $AB \times M \pm BC \times N = IN \times N$.

Q. E. D.

SECOND SOLUTION, *by* SCOTICUS.

Let A and B (fig. 11, pl. 1.) be the given points in the diameter, and C the centre of the given circle. Let us ſuppoſe the problem reſolved, or that AD, DB are drawn to a point D in the circumference, ſo that $M \times AD \pm N \times DB = M \times L$, a given ſpace. Join CD; then, by a well known propoſition in Geometry, (*ſee Math. Repoſitory, Vol.* I. *Page* 18, *Old Series*)

$AD^2 + \frac{CA}{BC} BD^2 = BA \times AC + \frac{BA}{BC} CD^2$. Now BA, AC, BC, CD are lines given in magnitude, therefore,

$AD^2 + \frac{CA}{BC} BD^2 =$ a given ſpace. Let us for the ſake of perſpicuity put $AD = x$ and $\frac{\sqrt{(CA \times CB)}}{CA} \times BD = y$, alſo $\frac{N \times \sqrt{(CB \times CA)}}{CA} - P$, where P is evidently a given line, then, to determine the magnitudes of x and y, we have

$$x^2 + y^2 = Q^2 \text{ a given ſpace}$$

$$x \pm \frac{P}{M} y = L \text{ a given line.}$$

And from theſe equations it is obvious how the values of x and y may be readily determined by various methods ſtrictly geometrical, but as theſe methods only ſerve to determine the magnitudes of the lines AD and BD without any regard to their poſition, it ſeems unneceſſary to point them out as they muſt be ſufficiently obvious to the reader.

THIRD SOLUTION, *by Mr.* JOHN JOHNSON, *Birmingham.*

Let A, B, (fig. 12, pl. 1.) be the given points in the diameter DE, and AC, BC the lines required to be drawn.

To the centre O draw the radius CO and make $M \times Q =$ to the given ſpace, then $AC \pm BC \times \frac{N}{M} = Q$, and by Prop. 6. of Dr. Stewart's General Theorems,

$$AC^2 + BC^2 \times \frac{OA}{OB} = BA \times AO + BA \times CO \times \frac{CO}{BO} = \text{to}$$

a given ſpace.

Hence the problem is reduced to this; to find two lines AC, BC ſuch, that the ſum or difference of AC and a line to which BC has a given ratio may be equal to a given line, and the ſquare of AC together with the ſpace to which the ſquare of BC has a given ratio may be equal to a given ſpace. To effect this, take (in fig. 13, pl. 1.) $Ia = Q$ and $(aR)^2 =$ to the given ſpace make RVT perpendicular to IR, and ſuch that $RV : IR = M : N$ and $RT^2 : RV^2 = OA : OB$; draw To parallel to the line joining IV and apply $ao = aR$; then if the perpendicular oc be demitted cutting IV produced at b; ac, cb are the lines required.

For, by ſimilar triangles $Ic : cb :: IR : RV :: N : M$, hence, $Ic = \frac{N}{M} \times cb$ and $ac + \frac{N}{M} \times cb = aI = Q =$ the given line. Again, by parallels $RV^2 : RT^2 :: cb^2 :: co^2 :: oB : oA$, therefore, $co = \frac{OA}{OB} \times cb^2$; conſequently $ac^2 + \frac{OA}{OB} \times cb^2 = ao^2 = AR^2 =$ the given ſpace. The letters to the left of oc are applicable to the ſum, and thoſe on the right to the difference.

Solutions were alſo received from Meſſrs. Butterworth *and* Smith.

X. QUESTION 10, *by Mr.* WILLIAM SMITH, *Liverpool.*

To determine the locus of the vertex of a triangle, the difference of whoſe ſides and the radius of whoſe inſcribed circle are given.

SOLUTION, *by Mr.* JOHN FAREY, No. 12, *Crown Street, Weſtminſter.*

Let LKG (fig. 14, pl. 1.) be the given inſcribed circle, and the indefinite line AB a tangent thereto at G, let GE be a diameter of the circle, and therefore perpendicular to AB; take GF = the given difference of the ſides, then through the points F and E draw the indefinite line FEC, which is the required *locus of the vertex.*

DEMONSTRATION. From any point C in the line EC, draw tangents to the circle, cutting AB in A and B, and touching the circle in L and K; then, ſince by the property of tangents CL = CK, AL = AG and BK = BG, the difference of AL and KB is the ſame as the difference of the ſides AC and CB, equal alſo to the difference of AG and BG, = FG by conſtruction. Biſect FG in M which will be the middle point of the baſe; then, becauſe GE = 2Gc, and FG = 2MG, Mc is parallel to FC, and FG = AC — CB.

Q. E. D.

COROLLARY. Hence in any plane triangle, having a circle inſcribed therein, and a diameter thereto drawn perpendicular to the baſe; if a right line be drawn from the vertex of the triangle through the top of the diameter of that circle it will cut off a ſegment of the baſe adjacent to the bottom of that diameter, or point of contact with the circle, equal to the difference of the ſides of the triangle.

SCHOLIUM. The diſcovery of the above locus preſents an eaſy method of conſtructing ſeveral caſes of triangles, ſome of which have not yet, to my knowledge, been conſtructed. In what follows I ſhall uſe the notation and arrangement of *Leybourn's Synopſis of Data for Conſtructing Triangles*, (ſee Page iv. of that Work) and earneſtly recommend to the correſpondents of the *Mathematical Repoſitory*, to adopt the ſame notation and arrangement of the *Data* in propoſing future problems or theorems relating to triangles.

In all the following conſtructions the given inſcribed circle, and a tangent thereto for the baſe of the triangle is to be firſt drawn as above, and a diameter EG perpendicular to the baſe; alſo the given difference of the ſides GF is to be ſet off on the baſe and biſected in M for the middle of the baſe, and the indefinite line FEC drawn for the locus of the vertex, then

1. $V.S - s.r.$ *(See No. 51 in the Synopſis.)*

Conſtruct a right-angled triangle, whoſe angle at the baſe is equal to $\frac{1}{2}V$ and its perpendicular or oppoſite ſide $= r$, (or if V is given in degrees, take a fourth proportional to *nat. ſine* $\frac{1}{2}V$, 1, and r) and with the hypothenuſe or this line as radius and centre c (the centre of the given inſcribed circle) deſcribe an arc cutting EC in C, draw CA and CB tangents to the inſcribed circle cutting AB in A and B, and ABC is the triangle required.

2. $B.S - s.r.$

From M the middle of B take MB and MA each equal to half B, and from A and B draw tangents to the inſcribed circle which will interſect in EC, in the point C (from what is done above) and form the required triangle ABC.

3. $-B+S+s.\ S-s.\ r.$

By the Student, page 34, cor. 2, half the firſt datum is equal to the ſegments of S and s next V, by the inſcribed circle, $=$ CL or CK; therefore, find the hypothenuſe of a right-angled triangle, whoſe legs are CK and Kc, and therewith proceed, as in the firſt caſe,

4. $P.S - s.r.$

Parallel to AB at the diſtance of P, draw a line cutting EC in C, from whence draw tangents, as in the 1ſt caſe, and ABC is the required triangle.

5. $S - s.\ S.r.$ *(No. 163 Synopsis.)*

Between the given lines GB and EC, draw a line $=$ S, to touch the given inſcribed circle; let it interſect GB in B and EC in C; from C draw a tangent to the inſcribed circle, interſecting GA in A, and ABC is the required triangle.

Note. If with $S - s$ there be given $S + s$, $S \times s$, $S : s$; $S^2 + s^2$, $S^2 - s^2$, &c. by known problems or *lineal ſections* S and s will be known, and the conſtruction as in the laſt caſe.

6. $S - s.\ A$ or $a.\ r.$

Conſtruct a right-angled triangle, whoſe angle at the baſe is equal to $\frac{1}{2}A$ or $\frac{1}{2}a$, and its perpendicular or oppoſite ſide $= r$, (or if A or a be given in degrees, take a fourth proportional to *nat. sine* $\frac{1}{2}A$, 1, and r) and with the hypothenuſe or this line as a radius and centre c deſcribe an arc cutting GB in B, from B draw a tangent to the inſcribed circle cutting EC in C, and from C draw another tangent to the ſame circle, cutting AB in A, and ABC is the triangle required.

7. $S - s.\ A - a.\ r.$ (*No.* 175 *Synopsis.*)

From c draw cC making with cE an angle equal $\frac{1}{2}(A - a)$, and interſecting EC in C (See Repoſitory, Vol. I. Page 284, Prop. 2, Old Series), then proceed as in caſe 1ſt to complete the triangle ABC.

8. $S - s.\ m$ or $n.\ r.$

Suppoſe the ſegment AI given, then becauſe AF $=$ GB and FG given, 2GI $+$ IB is given, or its half GZ, ſuppoſing IB biſected in Z. Draw ZQ perpendicular to AB; then becauſe it biſects IB it will alſo biſect BC in Q, and therefore the locus of the point Q will be a ſtraight line paſſing through the middle of FB and parallel to FC; conſequently the point Q is determined, and the conſtruction obvious.

9. $S - s.\ m - n.\ r.$ (*No.* 177 *Synopsis.*)

Take MI $= \frac{m - a}{2}$, and erect the perpendicular IC and complete the triangle as in the firſt caſe.

10. $S - s.\ m - n$, ſegs. by L. r.

Take MD $= \frac{m - n}{2}$, through c draw DcC, cutting EC in and complete the triangle as in the 1ſt caſe.

11. $S - s.\ m \times n$, ſegs. by $L + L^2.\ r.$

By Simpſon's Geometry III. 26, the ſecond datum is equal to $S \times s$, whence by *Lineal Sections*, the problem is reduced to the 5th caſe.

12. $S - s.\ m$ or n, ſegs. by inſcribed circle. r.

Take GB $= n$, and proceed as in the 6th caſe.

13. $S - s.\ l.\ r.$

On M as a centre, with the radius l (the line biſecting B) deſcribe an arc cutting EC in C, and proceed as in the 1ſt caſe.

14.

14. $S-s$. r. Diſtance of V from centre of the inſcribed circle.

This conſtruction is the ſame as the 1ſt caſe.

15. $S-s$. r. Neareſt diſtance of V from the inſcribed circle.

With radius $r+$ the given diſtance, and centre c deſcribe an arc cutting EC in C, and proceed as in the 1ſt caſe.

16. $S-s$. r. Diſtance of V from contact of B and inſcribed circle.

With the given diſtance as radius and G as centre deſcribe an arc cutting EC in C as before.

17. $S-s$. r. Diſtance of A or a from centre of inſcribed circle.

This is to be done as the 6th caſe.

18. $S-s$. r. Neareſt diſtance of A or a to inſcribed circle.

Proceed as in the 17th and 6th caſes.

19. $S-s$. r. Diſtance of centre of inſcribed circle from interſection of B and L.

With the given diſtance as radius and centre c, deſcribe an arc cutting AB in D, and proceed as in the 10th caſe.

20. $S-s$. r. Diſtance of centre of inſcribed circle from foot of P.

With the given diſtance as radius and centre c deſcribe an arc cutting AB in I, erect the perpendicular IC meeting the locus in C, and proceed as in the 1ſt caſe.

21. $S-s$. r. Diſtance of centres of inſcribed and circumſcribed circles.

The diameter of the circumſcribing circle becomes known from Prop. V. Art. II. Vol. II. of the Old Series, and the conſtruction from thence obvious.

The known *loci*, when the following pairs of data are given, viz. V and B, V and $B+S+s$, V and $-B+S+s$, V and inſc. □ on B, B and $S:s$, B and S^2+s^2, B and S^2-s^2, $S:n$ and L, $A-a$ and $m-n$, and others, preſent each a ſimilar ſeries of conſtructions, obvious to the geometer almoſt at firſt ſight; many of which, from the *limits* of one of the data being ſhewn, offer a much greater number of caſes than are given above. I beg leave to recommend to your ingenious correſpondents, a ſearch to diſcover as many of theſe *loci* as poſſible, even though they happen to be conic ſections, or even curves of higher orders; theſe inveſtigations appearing to offer one of the moſt pleaſing, and certainly not the leaſt uſeful applications of Algebra to Geometry.

The Solutions given by Meſſrs. Bazley, Butterworth, Dawes, Smith, *and* Wright, *were exactly the ſame as that given above by Mr.* Farey.

XI. QUESTION 11, *by Mr.* JOHN WRIGHT, *Norley*.

Given the baſe and vertical angle of a plane triangle to conſtruct it, when the ſolid of the line biſecting the vertical angle into the ſquare of the difference of the ſegments of the baſe made by the perpendicular is a maximum.

FIRST SOLUTION, *by Mr.* JOHN WRIGHT, *the Propoſer*.

CONSTRUCTION. Through the extremities B, H, (fig. 15, pl. 1.) of the given baſe deſcribe a circle BCHP ſuch, that the ſegment ACH may contain the given vertical angle; biſect BH in A with the diameter CP, in which determine the point S ſuch that, R being the centre of the circle, and $RI = CI$, the rectangle $SP \times SI$ may be equal to $AP \times RI$: draw SQ parallel to BH meeting the circle in Q, then joining BQ, HQ, BQH will be the triangle required.

DEMONSTRATION. Demit QD perpendicular to BH and DG to PQ; to the circle at Q draw the tangent TN terminating in BH and PC produced, and let TP be biſected in O. Then becauſe $TS : PS :: CS : RS$, by compoſition $TP = 2PO : PS :: CR = 2RI : RS$, or $PO : PO + OS :: RI : RS$, and by div. $PO : OS :: RI : IS :: SP : AP$ by conſtr. therefore, $OS \times PS = PO \times AP$; hence by my ſolution of Queſtion 298, No. XII. of the Old Series of the Repoſitory, the right-angled triangle FQD is greater than any other whoſe hypothenuſe paſſes through P, and having one leg parallel to TP, that can be formed within the triangle ATN, and much more then muſt it be greater than any other of a like conſtruction that can be formed within the ſegment HQCB. The triangle FQD being then a maximum, $FQ \times GD$, and conſequently $FQ \times GD \times CP$ muſt be a maximum; but $GD \times CP = FQ \times SQ$ by ſimilar triangles; conſequently $FQ^2 \times SQ$ or $FQ^2 \times (BD - DH)$ is a maximum.

Q. E. D.

SECOND

SECOND SOLUTION, *by Mr.* WILLIAM SMITH, *Liverpool.*

CONSTRUCTION. Let AB (fig. 16, pl. 2.) be the given baſe, AGBF the given circumſcribing circle, and GLMF its diameter, biſecting the baſe in M. On GM deſcribe a ſemicircle, to which draw a tangent PQN (the method of doing which is well known) meeting FG produced in Q and the tangent at F in P, ſo that QL may be equal to 4LF, draw LNC parallel to AB meeting the circle in C; join AC, CB and ACB will be the triangle required.

DEMONSTRATION. Join FC meeting AB in E, and demit the perpendicular CD. Then, by ſimilar triangles $EC \times CL^2 = FC \cdot CL \cdot ED$, but $CL \cdot ED = CD \cdot GL = ML \cdot GL = LN^2$; therefore, $EC \cdot CL^2 = FC \cdot LN^2$; but, ſince by conſtruction, $QL = 4LF$, the product $FL \cdot LN^4$ is a maximum, or the ſame multiplied into the conſtant quantity TG will be a maximum, that is $FC^2 \cdot LN^4$ is a maximum, or $FC \cdot LN^2$ is a maximum, that is, from what is done above, $EC \cdot CL^2$ is a maximum.

Q. E. D.

Ingenious Solutions were alſo received from Meſſrs. Bazley, Cavill, *and* Marrat.

XII. QUESTION 12, *by Mr.* JAMES CUNLIFFE, *Bolton.*

It is required to determine the *locus* of a point, from whence if perpendiculars be drawn to three ſtraight lines given by poſition, the ſum of the ſquares of the ſaid perpendiculars may be equal to a given ſpace.

FIRST SOLUTION, *by Mr.* J. I.

1. In the firſt place we ſhall conſider the propoſition in the caſe when the lines given by poſition meet in one point.

ANALYSIS. Let AB, AC, AD (fig. 17, pl. 1.) be three lines given by poſition interſecting one another in the point A, and let EB, EC, ED be the perpendiculars drawn to them from the point E. Join AE, and upon it as a diameter deſcribe a circle, which will paſs through D, C, B as is manifeſt. Join DC, CB, DB; then it is obvious that the ſeveral angles of the triangle DCB are given (being equal to the angles that the given lines make with one another) and therefore the triangle is given in ſpecies. Take any line pr (fig. 18, pl. 1.) and upon it conſtruct a triangle pqr ſimilar to DCB; about the triangle pqr deſcribe a circle of which m is the centre, and take n the centre of gravity of the three points pqr and join pm, qm, rm; pn, qn, rn and draw the diameter umns; alſo let O be the centre of the circle paſſing through D, C, B, and G the centre of gravity of the ſame points, and join DO, OB, OC; DG, BG, GC and draw the radius OGF and the lines AF and EF. Then becauſe the triangle pqr is ſimilar to the triangle DCB and that the points m and n are ſimilarly ſituated in reſpect of the one as O and G are in reſpect of the other, it may eaſily be proved that the ſeveral triangles pmn, qmn, rmn are ſimilar to the triangles DOG, BOG, COG; therefore, the ſeveral angles at O are equal to the ſeveral angles at m each to each; but the angles DAF, CAF, BAF are the halves of the angles at O; therefore, they are likewiſe the halves of the angles at m; conſequently the angles that AF makes with the given lines AB, AC, AD are given and the line AF is given by poſition.

Draw GH at right angles to OF and join OH, HF. Becauſe G is the centre of gravity of the three points D, C, B, it is a property very anciently known (Vide Cor. 1ſt. Prop 5th. Lib. 2d. Loc. Plan. a R. Simſon reſt.) that $DE^2 + EC^2 + EB^2 = DG^2 + BG^2 + CG^2 + 3GE^2$: And in like manner, $DO^2 + BO^2 + OC^2$, or $3OH^2 = DG^2 + BG^2 + CG^2 + 3GO^2$; therefore, $DG^2 + BG^2 + CG^2 = 3OH^2 - 3GO^2 = 3GH^2$, conſequently denoting by S the given ſpace that is equal to $DE^2 + EB^2 + EC^2$, there reſults $S = 3GH^2 + 3GE^2 = 3GH^2 + 3GF^2 + 3FE^2 = 3HF^2 + 3FE^2$: therefore, $\frac{S}{3} = HF^2 + FE^2$. Draw tn at right angles to ms and join mt, ts: then it is obvious that the triangles mtn, mts are ſimilar to HOG, HOF: therefore, 2tm or su : ts :: 2OD or AE : AF. Therefore, $\frac{S}{3} = \frac{ts^2}{su^2} \times AE^2 + EF^2$. Make $ts^2 : su^2 :: \frac{S}{3} : AK^2$, and su : ts :: AK : AL: then it will evidently follow, that,

$$AK^2 = AE^2 + \frac{AL^2}{AK^2} \times EF^2$$

a property of an ellipſe of which A is the centre, AK the ſemi-tranſverſe axis, and AL the focal diſtance. Therefore the *Locus* required is a given ellipſe.

CONSTRUCTION. Aſſume any line as pr; make the angle qpr = CAB and the angle qrp = CAD: about the triangle pqr deſcribe a circle of which m is the centre, and take n the centre of gravity of the points qpr, and draw the ſeveral lines as directed in the Analyſis. Draw the line AF to make the angles DAF, CAF, BAF equal to the halves of the angles tms, qms, rms: and make

$$ts^2 : su^2 :: \frac{S}{3} : AK^2$$

$$\text{and } su : ts :: AK : AL$$

then the locus required will be an ellipſe of which A is the centre, AK the ſemi-tranſverſe axis, and AL the focal diſtance.

2. We proceed to the caſe when the given lines do not interſect in one point.

ANALYSIS. Let the given lines form the triangle ABC (fig. 19, pl. 1.) by their mutual interſections; aſſume a point P within the triangle, and draw PQ, PR, PS perpendicular and PL, PK, PM parallel to AC, AB, and BC; then, if the perpendiculars drawn from the point E meet PL, PK, and PM in L, K, and M, it readily follows that, $EH^2 + EF^2 + EG^2$

$$= \left\{EL^2 + EM^2 + EK^2\right\} + \left\{PR^2 + PQ^2 + PS^2\right\} + 2\left\{EK \cdot PR + EL \cdot PQ - EM \cdot PS\right\}.$$

Again, twice the area of the triangle ABC is equal to $\left\{AC \cdot PR + BC \cdot PS + AB \cdot PQ\right\}$ and it is alſo equal to $\left\{AC \cdot EG + AB \cdot EF - BC \cdot EH\right\}$: therefore, taking equals from equals, $\left\{BC \cdot EK + AB \cdot EL - AC \cdot EM\right\} = 0$. Now, if we ſuppoſe the point P, which was aſſumed *ad libitum*, to be taken ſuch, that PR, PQ, PS are proportional to the given lines AC, BC, and AB, it is clear that,

$$\left\{PR \cdot EK + EL \cdot PQ - EM \cdot PS\right\} = 0.$$

Suppoſing then that the point P is taken as above, it will follow from what has already been ſhewn, that $EH^2 + EF^2 + EG^2 = \left\{EM^2 + EL^2 + EK^2\right\} + \left\{PR^2 + PQ^2 + PS^2\right\}$. Therefore, denoting by S the given ſpace that is equal to $EH^2 + EF^2$

+

$+ EG^2$, there reſults $EL^2 + EM^2 + EK^2 = S - \left\{PR^2 + PQ^2 + PS^2\right\}$. Therefore, ſuppoſing the point to be determined, this ſecond is manifeſtly reduced to the firſt caſe.

It remains only to determine the point P. Draw the lines APN, CPO, and BP. Then, as was ſuppoſed, $AB : AC :: PQ : PR$; therefore, $AB^2 : AC^2 :: AB \times PQ : AC \times PR :: \triangle APB : \triangle APC$. But it is eaſily ſhewn that $\triangle APB : \triangle APC :: \triangle ABN : \triangle ACN :: BN : NC$. Therefore, $AB^2 : AC^2 :: BN : NC$; therefore, the point N in which BC is cut in a given ratio is given, and the line AN is given by poſition. In like manner it is ſhewn that $BC^2 : CA^2 :: BO : AO$, and that CO is given by poſition; therefore the point P is determined.

By the preceding Analyſis, the ſecond caſe is reduced to the firſt. It follows too that the ſum of the ſquares of the perpendiculars drawn from the point is a minimum, or is leſs than the ſum of the ſquares of perpendiculars drawn from any other point; and hence, in this ſecond caſe, the locus is reſtricted, it being requiſite that the given ſpace ſhall not be greater than $\left\{PR^2 + PQ^2 + PS^2\right\}$.

Corollary. Although the caſe of three lines only has been conſidered, yet it is manifeſt that the ſame Analyſis and Conſtruction will apply, *mutatis mutandis*, whatever be the number of lines given by poſition.

SECOND SOLUTION, *by Mr.* WILLIAM SMITH, *Liverpool.*

Lemma. Let Y, W, t, &c. (fig. 20, pl. 1.) be any number of given points in a right line given in poſition, and let any line GP be drawn parallel to a line given in poſition and produced to P, ſo that a ſquare to which YG^2 has a given ratio, together with a ſquare to which WG^2 has another given ratio, together with a ſquare to which tG^2 has another given ratio, together with GP^2 may be equal to a given ſpace. Then I ſay the locus of the point P is a given ellipſe.

Demonstration. Take any point G and erect the perpendicular GE, ſuch that the triangle tGE may be equal to a ſquare to which tG^2 has a given ratio, and produce tE to meet YC, parallel to GE, in C; find the point L ſo that the triangle YGL may be equal to the ſquare to which YG^2 has the given ratio; find alſo the point R ſuch, that the triangle WGR may be equal to a ſquare to which WG^2 has the given ratio. Then, by hypotheſis, the $\triangle tGE + \triangle YGL + \triangle WGR + GP^2 =$ a given ſpace. Produce YL to meet tE in A, a given point, draw AD parallel to YC and join CD meeting GE in S; alſo produce WR to meet CD in B, a given point, and demit the perpendicular BF. Then the $\triangle tGE + \triangle YGL = \triangle tAY + \triangle AEL$, but YC and AD being parallel, it is evident that $GS = EL$, or the $\triangle AEL =$ the $\triangle DGS$, alſo that the $\triangle DGS + \triangle WGR = \triangle WBD + \triangle SBR$; draw WZ parallel to FB and join ZF meeting DL in O, then the $\triangle FGL = \triangle SBR$; therefore, the given ſum $(= \triangle tGE + \triangle YGL + \triangle WGR + GP^2) = \triangle YtA + \triangle WBD + \triangle FGL + GP^2$, but the triangles YtA and WBD are given; therefore, the $\triangle FGL + GP^2$ is given, that is, a ſquare to which FG^2 has a given ratio, together with GP^2 is given, and F a given point; therefore, by the well-known properties of the ellipse, the locus of P is a given ellipſe whoſe centre is F and ordinate GP, parallel to the conjugate of FG.

Analysis. Fig. 21, Pl. 1. Let AB, BC, CD, DA, &c. be any number of lines given in poſition, and P a point ſuch, that demitting the perpendiculars Pa, Pb, Pc, Pd, the ſum of their ſquares may be equal to a given ſpace. Produce one of the perpendiculars as Pa to meet AD in K, CD in I, and AB in H; at P erect an indefinite perpendicular Pirqpgfv, and take therein Pi ſuch, that $HP \cdot Pi = Pb^2$, and produce Hi to meet aL (drawn at half a right angle to aP) in R, demit the perpendicular RE, alſo draw HL parallel to RE meeting AR in L, and join EL meeting Pi in f and demit the perpendicular EO upon AB. Then, ſince $HP \cdot Pi = Pb^2$, by parallels, $HE \cdot Ea$ (ER) $= EO^2$; therefore, the locus of the point E is a right line BE given in poſition, which produce to meet DC in the given point X. Moreover, ſince the $\triangle HPi =$ half the ſquare on Pb, and the $\triangle aPv =$ half the ſquare on aP; therefore, half the ſum of the ſquares of Pb, Pa, is equal to half the ſum of the triangles HPi, $aPv = \triangle HRa + \triangle Riv$, but ſince HL is parallel to RE the $\triangle Riv = \triangle EPf$; therefore, half the ſquare on aP together with half the ſquare on bP $= \triangle HaR + \triangle EPf$. Again, find in Pi the point r, ſuch that $IP \cdot Pr$ may be $=$ to the ſquare of the perpendicular Pc demitted on IDC and join Ir meeting EL in S, demit the perpendicular SF, and draw IM parallel thereto, meeting EL in M, and join FM meeting Pr in g, and demit on CD the perpendicular Fv. Then ſince $IP \cdot Pr = Pc^2$; therefore $IF \cdot FS = Fy^2$; therefore the locus of the point F is a right line XF given in poſition, which produce to meet AD in Y, a given point; *alſo*, ſince the rect. $IPr = Pc^2$; therefore, half the ſum of the ſquares of Pa, Pb, Pc $= \triangle$ HaR $+ \triangle$ EPf $+ \triangle$ IPr $= \triangle$ HaR $+ \triangle$ IES $+ \triangle$ Srf, but ſince FS is parallel to IM, the $\triangle$ Srf $= \triangle$ FPg; therefore, half the ſum of the ſquares of Pa, Pb, Pc $= \triangle$ HaR $+ \triangle$ IES $+ \triangle$ FPg. Find alſo, in Pr the point p, ſuch, that KP·Pp may be = to Pd^2, join Kp and produce it to meet FM in T, demit the perpendicular TG and draw KN parallel thereto, meeting FM in N, and join GN meeting Pp in q and demit the perpendicular Gz on AD. Then ſince KP·Pp = Pd^2; therefore, KG·GT = Gz^2; therefore the locus of the point G is a right line YG, given in poſition. Moreover, ſince the $\triangle$ KPp = half the ſquare on Pd; therefore, half the ſum of the ſquares of Pa, Pb, Pc, Pd $= \triangle$ HaR $+ \triangle$ IES $+ \triangle$ KET $+ \triangle$ GPq, the ſum of theſe triangles is, therefore, equal to a given ſpace, but $\triangle$ HaR is to BE^2 in a given ratio, that is half the rectangle Ha·aE is to BE^2 in a given ratio, the $\triangle$ IES is to EX^2 in a given ratio, and the $\triangle$ KFT is to YG^2 in a given ratio; alſo drawing Bt and XW parallel to aG meeting YG in t and W; BE^2 is to Gt^2 in a given ratio, and EX^2 is to GW^2 in a given ratio; therefore, (the $\triangle$ HaR $+ \triangle$ IES $+ \triangle$ KFT $+ \triangle$ GPq =) a ſquare to which Gt^2 has a given ratio, together with a ſquare to which GW^2 has a given ratio, together with a ſquare to which YG^2 has a given ratio, together with a ſquare to which GP^2 has a given ratio, is given by the queſtion; and therefore, by the Lemma, the locus of the point P is a given ellipſe.

XIII. QUESTION 13, *by* SCOTICUS.

Let φ and ψ be two arches ſuch that ſin. φ + cos. $\psi = a$, and ſin. ψ + cos. $\varphi = b$: Required ſin φ and cos. φ, alſo ſin. ψ and cos. ψ?

FIRST SOLUTION, *by the Propoſer.*

Put $\theta = 90° - \psi$, then coſ. ψ = ſin. θ and ſin. ψ = coſ. θ, thus the propoſed equations become

$$\text{ſin. } \varphi + \text{ſin. } \theta = a$$
$$\text{coſ. } \varphi + \text{coſ. } \theta = b.$$

Now, from the common theorems for finding the ſine and coſine of the ſum and difference of two arches, (ſee Emerſon's Trig. Book I. Prop. 6. Schol. 2.) we have

$$\text{ſin.}\,\varphi + \text{ſin.}\,\theta = 2\,\text{ſin.}\,\tfrac{1}{2}(\varphi + \theta)\,\text{coſ.}\,\tfrac{1}{2}(\varphi - \theta)$$
$$\text{coſ.}\,\varphi + \text{coſ.}\,\theta = 2\,\text{coſ.}\,\tfrac{1}{2}(\varphi + \theta)\,\text{coſ.}\,\tfrac{1}{2}(\varphi - \theta)$$

therefore,

$$2\,\text{ſin.}\,\tfrac{1}{2}(\varphi + \theta)\,\text{coſ.}\,\tfrac{1}{2}(\varphi - \theta) = a$$
$$2\,\text{coſ.}\,\tfrac{1}{2}(\varphi + \theta)\,\text{coſ.}\,\tfrac{1}{2}(\varphi - \theta) = b.$$

Taking now the ſum of the ſquares of both equations, and obſerving that $\text{coſ.}^2\,\tfrac{1}{2}(\varphi + \theta) + \text{ſin.}^2\,\tfrac{1}{2}(\varphi + \theta) = 1$ we find

$$4\,\text{coſ.}^2\,\tfrac{1}{2}(\varphi - \theta) = a^2 + b^2.$$

Hence we have immediately

$$2\,\text{ſin.}\,\tfrac{1}{2}(\varphi - \theta) = \sqrt{(4 - a^2 - b^2)},$$

and from the ſame two equations

$$\text{coſ.}\,\tfrac{1}{2}(\varphi + \theta) = \frac{b}{\sqrt{(a^2 + b^2)}},\quad \text{ſin.}\,\tfrac{1}{2}(\varphi + \theta) = \frac{a}{\sqrt{(a^2 + b^2)}}.$$

But by the Prop. in Emerſon's Trigonometry already quoted,

$$\text{ſin.}\,\varphi - \text{ſin.}\,\theta = 2\,\text{coſ.}\,\tfrac{1}{2}(\varphi + \theta)\,\text{ſin.}\,\tfrac{1}{2}(\varphi - \theta)$$
$$\text{coſ.}\,\varphi - \text{coſ.}\,\theta = -2\,\text{ſin.}\,\tfrac{1}{2}(\varphi + \theta)\,\text{ſin.}\,\tfrac{1}{2}(\varphi - \theta).$$

Therefore,

$$\text{ſin.}\,\varphi - \text{ſin.}\,\theta = \frac{b\sqrt{(4 - a^2 - b^2)}}{\sqrt{(a^2 + b^2)}}$$
$$\text{coſ.}\,\varphi - \text{coſ.}\,\theta = \frac{a\sqrt{(4 - a^2 - b^2)}}{\sqrt{(a^2 + b^2)}}.$$

Hence, and from the given equations, we derive the following expreſſions, in which $n = \frac{\sqrt{(4 - a^2 - b^2)}}{\sqrt{(a^2 + b^2)}} = \sqrt{(\frac{4}{a^2 + b^2} - 1)}$.

$$\text{ſin.}\,\varphi = \tfrac{1}{2}(a + bn)$$
$$\text{coſ.}\,\varphi = \tfrac{1}{2}(b - an)$$
$$\text{coſ.}\,\theta = \text{ſin.}\,\psi = \tfrac{1}{2}(b + an)$$
$$\text{ſin.}\,\theta = \text{coſ.}\,\psi = \tfrac{1}{2}(a - bn).$$

SECOND SOLUTION, *by the Rev. Mr.* SCURR.

Let the ſines of φ and ψ be x and y reſpectively; then will the coſines be denoted by $\sqrt{(1 - x^2)}$ and $\sqrt{(1 - y^2)}$.

By the queſtion $x + \sqrt{(1 - y^2)} = a$,

and $y + \sqrt{(1 - x^2)} = b$:

From the firſt equation $x = a - \sqrt{(1 - y^2)}$, and from the ſecond $x = \sqrt{\left\{1 - (b + y)^2\right\}}$.

Hence, $a - \sqrt{(1 - y^2)} = \sqrt{\left\{1 - (b + y^2)\right\}}$:

This equation by proper reduction becomes,

$$y^2 - by = \frac{a^2}{s^2} - \frac{s^2}{4}, \text{ where } s^2 = a^2 + b^2:$$

$$\text{Whence } y = \tfrac{1}{2}b - \sqrt{\frac{4a^2 - s^4 \times b^2 s^2}{4s^2}} = \tfrac{1}{2}b - \tfrac{1}{2}a\sqrt{\left(\frac{4}{a^2 + b^2} - 1\right)}.$$

Every thing required by the queſtion may now be very eaſily found.

THIRD SOLUTION, *by Mr.* WM. SMITH, *Liverpool.*

LEMMA. Let ZN, NW, WZ (fig. 22, pl. 1.) be three right lines given in poſition, and let O be a given point; it is required to find the point S, in WZ ſuch that drawing SE, SC parallel to lines given in poſition, and meeting ZN, NW in E and C, the lines drawn from O to the points E and C may be equal to each other, that is, ſo that OE may be equal OC.

Demit the perpendiculars OF, OD, and draw FP, DL parallel to SE, SC; then, ſince OE = OC it is plain that $DC^2 - EF^2$ $(= OF^2 - OD^2)$ is given, but DC is to LS, and EF to PS in given ratios; therefore, PL being a given line, the problem is reduced to the finding the point S therein, ſuch that a ſquare to which SL^2 has a given ratio, *minus* a ſquare to which SP^2 has another given ratio, may be equal to a given ſpace, the method of doing which is very ſimple. Hence the following conſtruction of the problem propoſed by Scoticus.

Take (fig. 23, pl. 1.) OG, OM equal to *a* and *b* reſpectively, and biſect them in I and L, erect the indefinite perpendicular OW, and take OH, OQ equal to OI, OL reſpectively; draw QF, HT parallel to OM meeting IF and LT, drawn parallel to OW, in F and T. Take HW ſo that HW may be to HT as MO to OG, alſo take QZ ſuch that QZ may be to QF as GO to OM; join WT, ZF and produce them to meet OM in K and N. Then, by the *Lemma*, find the point S in GV, drawn at half a right angle to GO, ſuch that drawing SC parallel to OM, meeting WK in C, and SE parallel to WO meeting ZN in E, the lines drawn from O to E and C may be equal; join OE, OC and from the centre O with radius OE or OC deſcribe a circle meeting OM in A, ſo ſhall AC, AE be the arches required.

DEMONSTRATION. Demit the perpendiculars ED, CB *meeting QF, HT in R and P.* Then, ſince, by conſtruction, OGS is half a right angle, GD (= DS =) = CB, and, by conſtruction, BP (OH) = GI (OI, therefore, ID = PC : but PC : BL (PT) (as HW : HT) :: MO : OG, that is, ID : BL :: MO : OG; therefore, ID × 2OG = BL × 2MO, that is, $GD^2 - DO^2 = MB^2 - OB^2$: but, by the circle $OD^2 + DE^2 = OB^2 + BC^2$ (DG^2), therefore, (BC^2) $GD^2 - DO^2 = DE^2 - OB^2$, and, therefore, $MB^2 - BO^2 = DE^2 - BO^2$, or MB = DE. But, by conſtruction, MB + BO (= DE + BO) = b, and GD + DO = CB + DO = a; therefore, AC (φ) and AE (ψ) are two arches, ſuch that ſin. φ + coſ. $\psi = a$, and ſin. ψ + coſ. $\varphi = b$. Conſequently AC, AE are the two arches required.

Solutions were alſo received from Meſſrs. Dawes *and* Marrat.

XIV. QUESTION 14, *by* HYPATIA.

Required the value of $\left(1 + \frac{1}{n}\right)^n$ when n is indefinitely great.

FIRST SOLUTION, *by Mr.* THOMAS BAZLEY, *Bolton.*

Let the given expreſſion $(1 + \frac{1}{n})^n$ be actually involved by the Binomial Theorem, and we obtain

$$1 + n.\frac{1}{n} + n.\frac{n-1}{1}.\frac{1}{n^2} + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{1}{n^3} + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{1}{n^4} + \&c.$$

but ſince n is indefinitely great, the factors n, $n - 1$, $n - 2$, &c. are equal to each other and to n, hence it becomes

$$1 + \frac{n}{n} + \frac{n^2}{2n^2} + \frac{n^3}{2.3n^3} + \frac{n^4}{2.3.4n^4} + \&c. = 1 + 1 + \frac{1}{2} + \frac{2}{2.3} + \frac{1}{2.3.4} + \&c;$$

which ſeries (it is well known) expreſſes the number 2·718282, whoſe hyp. log. is unity.

SECOND SOLUTION, *by* SCOTICUS.

Let $x = (1 + \frac{1}{n})^n$, then taking the hyp. log. of both ſides of *the equation*

log.

$$\log. x = n \times \log. (1 + \frac{1}{n})$$

but $\log. (1 + \frac{1}{n}) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \&c.$

therefore, $\log. x = 1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \&c.$

and, when n is indefinitely great, $\log. x = 1$. Hence $x =$ the number whoſe hyp. log. is $1 = 2{\cdot}7182818$, the anſwer.

THIRD SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

Let $v = (1 + \frac{x}{n})^n$, then taking the fluxions,

$$\dot{v} = \dot{x}\,(1 + \frac{x}{n})^{n-1};$$

let the latter equation be divided by the former, and we have

$$\frac{\dot{v}}{v} = \frac{\dot{x}}{1 + \frac{x}{n}}$$

which equation, when n is indefinitely great becomes

$$\frac{\dot{v}}{v} = \dot{x}$$

Hence, taking the fluents, $v = e^x$, where $e = 2{\cdot}7182818 =$ the number whoſe hyp. log. is unity, and when $x = 1$, then $v = e$.

XV. QUESTION 15, *by* ANALYTICUS.

Required a general expreſſion for the area of a curve whoſe equation is

$$y\sqrt[3]{(1 - 3x^2)} = \frac{a}{3 - x^2}.$$

SOLUTION, *by* ANALYTICUS, *the Propoſer.*

From the given equation we have the fluxion of the area

$$= \frac{a}{3 - x^2} \times \frac{\dot{x}}{\sqrt[3]{(1 - 3x^2)}}.$$

Aſſume $\sqrt[3]{(1-3x^2)} = 1 - \frac{x}{z}$, hence, $x = \frac{3z}{2}(1+z^2) - \frac{3z}{2}\sqrt{(z^4+2z^2-\frac{1}{3})}$, ſo that by taking the fluxion and ſubſtituting, &c. the fluxion of the area is transformed to

$$\frac{\frac{1}{2}a(1-z^2)\dot{z}}{(1+z^2)(3z^2-1)} - \frac{a}{2\sqrt{(3)}} \cdot \frac{3z^4+1}{3z^4+2z^2-1} \cdot \frac{\dot{z}}{\sqrt{(3z^4+6z^2-1)}}$$

And in this expreſſion the firſt part of the fluxion is rational.

To transform the ſecond part, let us further aſſume $\sqrt{(3z^4+6z^3-1)} = vz$, hence the ſecond term becomes

$- \frac{a}{2\sqrt{(3)}} \cdot \frac{\dot{v}}{v^2-4}$, thus upon the whole the fluxion of the area is transformed to

$$\frac{\frac{1}{2}a(1-z^2)\dot{z}}{(1+z^2)(3z^2-1)} - \frac{a}{2\sqrt{3}} \cdot \frac{\dot{v}}{v^2-4}$$

an expreſſion entirely without radicals, and therefore integrable by methods ſufficiently known.

XVI. QUESTION 16, *by* AMICUS.

It is required to find the ſurface of an oblique cone the radius of the baſe being unity, the perpendicular from the vertex $\frac{1}{2}$ and the diſtance of that perpendicular from the centre of the baſe $\frac{1}{2}\sqrt{3}$.

FIRST SOLUTION, *by* X.

Let C (fig. 24, pl. 1.) be the centre of the baſe, V the vertex, VA the perpendicular from the vertex, meeting the baſe in A: join CA meeting the circumference of the baſe in H, take B any point in the circumference, and draw the tangent BD, meeting CA produced in E, draw VD perpendicular to BE, and join CB, BA, DA, VE.

The triangles VDE, VAE, VAD are right-angled at D and A, therefore, $VE^2 - EA^2 = VD^2 - DA^2$, hence $AE^2 - AD^2 = VE^2 - VD^2 = DE^2$, therefore, AD is perpendicular to EB, and *conſequently* parallel to CB.

By hypotheſis CB, the radius of the baſe, $= 1$.

Put CA, the diſt. of the perp. from the centre $(= \frac{1}{2}\surd 3) = d$,
VA the perp. from the vertex $(= \frac{1}{2}) = p$,
HB, any variable arch of the circumference $= \varphi$.
The triangles ECB, EAD are ſimilar, hence
EC : CB :: EA : AD,

or $\text{ſec.}\,\varphi : 1 :: \text{ſec.}\,\varphi - d : \text{AD} = \frac{\text{ſec.}\,\varphi - d}{\text{ſec.}\,\varphi} = 1 - d\,\text{coſ.}\,\varphi$

Therefore, $\text{DV} = \surd\left\{p^2 + (1 - d\,\text{coſ.}\,\varphi)^2\right\}$; now the fluxion of the ſurface, comprehended by the arch φ, and lines drawn from the vertex to the extremities of that arch, is equal to $\frac{1}{2}\dot{\varphi} \times \text{VD}$, therefore, putting s for that ſurface,

$$\dot{s} = \tfrac{1}{2}\dot{\varphi}\,\surd\left\{p^2 + (1 - d\,\text{coſ.}\,\varphi)^2\right\}.$$

To prepare this fluxion for integration let us aſſume another arch ψ, ſo related to φ that

$\text{coſ.}\,\varphi = \frac{m + \text{coſ.}\,\psi}{1 + m\,\text{coſ.}\,\psi}$, or $\text{tan.}\,\frac{1}{2}\varphi = \surd(\frac{1-m}{1+m})\,\text{tan.}\,\frac{1}{2}\psi$,

where it appears that while φ increaſes from o to π (or 180°) ψ alſo increaſes from o to π.

From the aſſumed equation we find

$\dot{\varphi} = \frac{1}{\text{ſin.}\,\varphi}\cdot\frac{(1-m^2)\dot{\psi}\,\text{ſin.}\,\psi}{(1+m\,\text{coſ.}\,\psi)^2}$, and $\text{ſin.}\,\varphi = \frac{\surd(1-m^2)\,\text{ſin.}\,\psi}{1+m\,\text{coſ.}\,\psi}$,

and, therefore, $\dot{\varphi} = \surd(1-m^2)\cdot\frac{\dot{\psi}}{1+m\,\text{coſ.}\,\psi}$;

we next ſubſtitute $\frac{m + \text{coſ.}\,\psi}{1 + m\,\text{coſ.}\,\psi}$ for $\text{coſ.}\,\varphi$ in the radical quantity $\surd\left\{p^2 + (1 - d\,\text{coſ.}\,\varphi)^2\right\}$ which thus becomes

$\frac{\surd\alpha + \beta\,\text{coſ.}\,\psi + \gamma\,\text{coſ.}^2\,\psi)}{1 + m\,\text{coſ.}\,\psi}$ where

$$\alpha = p^2 + (1 - md)^2$$
$$\beta = 2\left\{p^2 m + (m - d)(1 - md)\right\},$$
$$\gamma = m^2 p^2 + (d - m)^2.$$

Aſſume $\beta = 0$; then, by reſolving a quadratic equation, we find $m = \frac{1 + p^2 + d^2}{2d} - \surd\left\{\left(\frac{1 + p^2 + d^2}{2d}\right)^2 - 1\right\}$,

where the leaſt root is taken, in order that m may be leſs than *unity, and thus* ψ *a real quantity.*

The radical $\sqrt{(\alpha + \beta \text{ cof. } \psi + \gamma \text{ cof.}^2 \psi)}$ may now be ex-preſſed thus $\sqrt{(\alpha + \gamma)} \times \sqrt{(1 - \frac{\gamma}{\alpha + \gamma} \text{ ſin.}^2 \psi)}$: now from the preceding equations we have

$$\alpha + \gamma = (1 + m^2)\, p^2 + (1 - md)^2 + (d - m)^2$$
$$mp^2 = -(m - d)(1 - md)$$

hence, by exterminating p^2 we have more ſimply

$$\alpha + \gamma = (1 - m^2)^2 \frac{d}{m}.$$

Again, from theſe equations

$$\alpha = p^2 + (1 - md)^2$$
$$m^2 p^4 = (d - m)^2 (1 - md)^2$$
$$\gamma = m^2 p^2 + (d - m)^2$$

by exterminating $m^2 p^4$ and $1 - md$ we find

$\alpha\,(d - m)^2 = \gamma p^2$, and hence $\frac{\gamma}{\alpha + \gamma} = \frac{(d - m)^2}{p^2 + (d - m)^2}$,

let this value of $\frac{\gamma}{\alpha + \gamma}$ be denoted by e^2, and our original fluxionary equation $\dot{s} = \frac{1}{2}\dot{\varphi} \sqrt{\left\{p^2 + (1 - d \text{ cof. } \varphi)^2\right\}}$ is transformed to

$$\dot{s} = \tfrac{1}{2}(1 - m^2)^{\frac{3}{2}} \sqrt{(\tfrac{d}{m})}\ \frac{\dot{\psi} \sqrt{(1 - e^2 \text{ ſin.}^2 \psi)}}{(1 + m \text{ cof. } \psi)^2}.$$

We are next to reduce this new fluxionary expreſſion to others more ſimple, whoſe fluents are commonly known. For this purpoſe we aſſume the algebraic expreſſion

$\frac{\text{ſin. } \psi \sqrt{(1 - e^2 \text{ ſin.}^2 \psi)}}{1 + m \text{ cof. } \psi}$, the fluxion of which is

$$\frac{\dot{\psi} \text{ cof. } \psi\, \Delta}{1 + m \text{ cof. } \psi} - \frac{e^2 \dot{\psi} \text{ ſin.}^2 \psi \text{ cof. } \psi}{(1 + m \text{ cof. } \psi)\, \Delta} + \frac{m \dot{\psi} \text{ ſin.}^2 \psi\, \Delta}{(1 + m \text{ cof. } \psi)^2}$$

where, for the ſake of brevity, Δ is put for the radical $\sqrt{(1 - e^2 \text{ ſin.}^2 \psi)}$; and here it is evident that this fluxionary expreſſion, the fluent of which we already know, involves in its terms the fluxion $\frac{\dot{\psi}\, \Delta}{(1 + m \text{ cof. } \psi)^2}$ whoſe fluent we are inveſtigating. That we may the more readily ſeparate this complex fluxion into its component terms, it will be convenient, for the preſent, to ſubſtitute x, and $1 - x^2$ for cof. ψ, and ſin.^{2} ψ reſpectively, and then reduce the whole to a common denominator, it thus becomes

ma

$$\frac{me^2x^4 + 2e^2x^3 + (1-2e^2)x + m(1-e^2)}{(1+mx)^2} \cdot \frac{\dot{\psi}}{\Delta},$$

and this again by methods of reduction ſufficiently obvious is reſolvable into

$$\left\{\frac{-e^2}{m}(1-x^2) + \frac{1}{m(1+mx)} - \frac{1-m^2}{m} \cdot \frac{1-e^2(1-x^2)}{(1+mx)^2}\right\} \frac{\dot{\psi}}{\Delta}:$$

now the laſt term of this expreſſion is evidently the very fluxion propoſed, which, it thus appears, depends upon the fluxion of an algebraic quantity, and other fluxions, whoſe fluents are to be farther the ſubject of inveſtigation.

Let us now put $\frac{1 - m\,\text{coſ.}\,\psi}{1 - m^2\,\text{coſ.}^2\,\psi}$ for $\frac{1}{1 + m\,\text{coſ.}\,\psi}$ and reſtore coſ. ψ, and ſin.^{2} ψ for x and $1 - x^2$, then putting $\dot{Z}$ for the whole expreſſion, or for the fluxion of $\frac{\text{ſin.}\,\psi\Delta}{1 + \text{coſ.}\,\psi}$ we have by tranſpoſition $(1 - m^2)\,\frac{\dot{\psi}\sqrt{(1 - e^2\,\text{ſin.}^2\,\psi)}}{(1 + m\,\text{coſ.}\,\psi)^2}$ equal to

$$-m\dot{Z} - \frac{m\dot{\psi}\,\text{coſ.}\,\psi}{(1-m^2\,\text{coſ.}^2\psi)\Delta} + \frac{\dot{\psi}}{(1-m^2\,\text{coſ.}^2\psi)\Delta} - \frac{e^2\,\dot{\psi}\,\text{ſin.}^2\,\psi}{\Delta}$$

The fluent of $\dot{Z}$ we already know; the ſecond term $\frac{m\dot{\psi}\,\text{coſ.}\,\psi}{(1 - m^2\,\text{coſ.}^2\,\psi)\Delta}$ may be integrated by an arch of a circle; the two remaining terms, by putting $\frac{m^2}{1 - m^2} = n$ and ſubſtituting $1 -$ ſin.^{2} ψ for coſ.^{2} ψ, may be reduced to this form

$$\dot{\psi}\Delta - \frac{1-n}{2} \cdot \frac{\dot{\psi}}{\Delta} + \frac{1+n}{2} \cdot \frac{1 - n\,\text{ſin.}^2\,\psi}{1 + n\,\text{ſin.}^2\,\psi} \cdot \frac{\dot{\psi}}{\Delta}$$

where $\dot{\psi}\Delta$ and $\frac{1-n}{2} \cdot \frac{\dot{\psi}}{\Delta}$ are integrable by the arcs of conic ſections: as to the fluxion $\frac{1 - n\,\text{ſin.}^2\,\psi}{1 + n\,\text{ſin.}^2\,\psi} \cdot \frac{\dot{\psi}}{\Delta}$ its fluent is in general a tranſcendant of a higher order than can be expreſſed by arches of conic ſections; there are caſes, however, in which it may be expreſſed by an arch of a circle, and this happens in particular when $n = e$: Now, by the queſtion, $d = \frac{1}{2}\sqrt{(3)}$, $p = \frac{1}{2}$, and, therefore, $m = \frac{1}{3}\sqrt{(3)}$, $e = \frac{1}{2}$, and $n = \frac{1}{2} = e$, hence it appears, that the ſimplification juſt now mentioned, takes place in the caſe under conſideration.

Upon the whole it appears that, by ſubſtituting numbers for p, d, m, and e,

$$\dot{s}=\tfrac{1}{2}\left\{\frac{-1}{\sqrt{(3)}}\dot{Z}-\sqrt{3}\,\frac{\dot{\psi}\,\text{coſ.}\psi}{(3-\text{coſ.}^2\psi)\Delta}+\dot{\psi}\Delta-\tfrac{1}{4}.\frac{\dot{\psi}}{\Delta}+\tfrac{1}{4}.\frac{1-\frac{1}{2}\text{ſin.}^2\psi}{1+\frac{1}{2}\text{ſin.}^2\psi}.\frac{\dot{\psi}}{\Delta}\right\}$$

where $\Delta=\sqrt{(1-\tfrac{1}{4}\,\text{ſin.}^2\,\psi)}$.

Now the fluent of $\dot{Z}$ is $\dfrac{\text{ſin.}\,\psi\Delta}{1+m\,\text{coſ.}\,\psi}$.

Take θ an arch of a circle ſuch that tang. $\theta=\tfrac{1}{2}\sqrt{(3)}\dfrac{\text{ſin.}\,\psi}{\Delta}$,

then, the fluent of $\sqrt{3}\,\dfrac{\dot{\psi}\,\text{coſ.}\,\psi}{(3-\text{coſ.}^2\psi)\,\Delta}$ is the arch θ.

The fluent of $\dot{\psi}\Delta$ or $\dot{\psi}\,\sqrt{(1-\tfrac{1}{4}\,\text{ſin.}^2\,\psi)}$ is evidently an arch of an ellipſe, having its semi-conjugate axis $=1$, and excentricity $=e=\tfrac{1}{2}$, the arch to be reckoned from the extremity of the conjugate axis to that of an ordinate to the tranſverſe, whoſe diſtance from the centre is ſin. ψ; let this arch be denoted by E.

Let ε denote the excentricity of another ellipſe, ſuch, that the ſemi-conjugate axis being alſo 1, $\varepsilon=\dfrac{1-\sqrt{(1-e^2)}}{1+\sqrt{(1-e^2)}}=\dfrac{1-\frac{1}{2}\sqrt{(3)}}{1+\frac{1}{2}\sqrt{(3)}}$, and ω an arch of a circle, ſuch that ſin. $\omega=$ $\dfrac{\left\{1+\sqrt{(1-e^2)}\right\}\,\text{ſin.}\,\psi\,\text{coſ.}\,\psi}{\sqrt{(1-e^2\,\text{ſin.}^2\,\psi)}}$, or ſo that tan. $(\omega-\psi)=\sqrt{(1-e^2)}.\,\text{tan.}\,\psi$, where it is to be obſerved that while ψ increaſes from 0 to π, ω increaſes from 0 to 2π, let E′ denote an arch of this ellipſe, intercepted between the conjugate axis, and an ordinate to the tranſverſe, the diſtance of the ordinate from the centre being $=$ ſin. ω, then, putting c $(=\tfrac{1}{2}\sqrt{3})$ for the ſemi-conjugate of the former ellipſe, the fluent of $\dfrac{\dot{\psi}}{\Delta}$ or

$$\frac{\dot{\psi}}{\sqrt{(1-e^2\,\text{ſin.}^2\,\psi)}}\text{ is }\frac{1}{c}\left\{-\text{E}+\frac{1+c}{2}\,\text{E}'+\frac{1-c}{2}\,\text{ſin.}\,\omega\right\}^{*}$$

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* See *Traité du Calcul Differentiel et du Calcul Integral par* La Croix, Vol. II, *page 181; the ſame may be found from* Landen's *Memoirs.*

Finally, let τ be ſuch an arch that $\frac{(1 + e)\,\text{ſin.}\,\psi}{1 + e\,\text{ſin.}^2\,\psi}$ or

$\frac{3\,\text{ſin.}\,\psi}{2 + \text{ſin.}^2\,\psi} = \text{ſin.}\,\tau$, then the fluent of

$\frac{1 - \frac{1}{2}\,\text{ſin.}^2\,\psi}{1 + \frac{1}{2}\,\text{ſin.}^2\,\psi} \cdot \frac{\dot\psi}{\sqrt{(1 - \frac{1}{4}\,\text{ſin.}^2\,\psi)}}$ is equal to $\frac{2}{3}\tau$
and theſe fluents being ſubſtituted in the general expreſſion will give the area required.

The whole area, or the fluent generated, while φ from o becomes 2π, will be found by conſidering that ψ and τ alſo increaſe from o to 2π, θ increaſes from o to a certain limit, and then decreaſes to o, and ω increaſes from o to 4π; hence $Z = 0$, $\theta = 0$, $\tau = 2\pi$, the ellipſe whoſe excentricity is e increaſes from o to the whole circumference, and the ellipſe whoſe excentricity is ε increaſes from o to two circumferences; hence, putting A for a quadrant of the former ellipſe, and B for a quadrant of the latter, the whole area is

$$\tfrac{1}{2}\pi + (2 + \frac{1}{2c})\,A - (\tfrac{1}{2} + \frac{1}{2c})\,B = \tfrac{1}{2}\pi + (2 + \frac{1}{\sqrt{3}})\,A - (\tfrac{1}{2} + \frac{1}{\sqrt{3}})\,B$$

SECOND SOLUTION, *by Mr.* J. I.

Let r denote the radius of the baſe of the cone, and h the perpendicular drawn from the vertex to the plane of the baſe: Let z denote a variable arch of the circular baſe reckoned from a fixt origin, and let ρ denote the line drawn in the conical ſurface from the extremity of z, to the vertex: From the extremity of the indefinitely ſmall arch $\dot z$, draw a perpendicular to ρ; then that perpendicular will be one leg of a right-angled triangle, of which $\dot z$ is the hypothenuſe, and $\dot\rho$ the other leg: it will, therefore, be equal to $\sqrt{(\dot z^2 - \dot\rho^2)}$. Therefore, the fluxion of the conical ſurface bounded by ρ, or the area of the ſmall triangle ſtanding on the baſe $\dot z$, and bounded by lines drawn to the vertex of the cone, will be equal to $\frac{\rho}{2}\sqrt{(\dot z^2 - \dot\rho^2)}$. If now we put s for the conical ſurface ſought, we have

$$\dot s = \frac{\rho}{2}\sqrt{(\dot z^2 - \dot\rho^2)}.$$

Draw a ſtraight line from the foot of the perpendicular through the centre of the baſe, and let a, denote the part of it intercepted between the perpendicular and the centre of the baſe: Let φ be the meaſure of z on the circle having unity for radius, and ſuppoſe the origin of z to be firſt in the extremity of the diameter drawn through the foot of the perpendicular: Then, we readily get $\rho^2 = h^2 + (a - r\,\text{coſ.}\,\varphi)^2 + r^2\,\text{ſin.}^2\,\varphi = h^2 + a^2 + r^2 - 2ar\,\text{coſ.}\,\varphi$: hence $\dot{\rho} = r\dot{\varphi} \times \frac{a\,\text{ſin.}\,\varphi}{\rho}$: But $\dot{z} = r\dot{\varphi}$: therefore, by ſubſtitution $\dot{s} = \frac{r\dot{\varphi}}{2}\sqrt{(\rho^2 - a^2\,\text{ſin.}^2\,\varphi)} = \frac{r\dot{\varphi}}{2} \times \sqrt{(h^2 + (r - a\,\text{coſ.}\,\varphi)^2.)}$ And putting $m = \frac{r}{h}$, $n = \frac{a}{h}$ we get

$$\dot{s} = \frac{rh}{2} \times \dot{\varphi}\sqrt{(1 + (m - n\,\text{coſ.}\,\varphi)^2.)}$$

Put now $y = m - n\,\text{coſ.}\,\varphi$; then $n\,\text{ſin.}\,\varphi = \sqrt{(n^2 - (m - y)^2)}$ And $\dot{\varphi} = \frac{\dot{y}}{\sqrt{(n^2 - (m - y)^2)}}$: therefore,

$$\frac{2\dot{s}}{rh} = \frac{\dot{y}\sqrt{(1 + y^2)}}{\sqrt{(n^2 - (m - y)^2)}}$$

It would far exceed our limits to treat the fluxional expreſſion juſt obtained in a general manner: we ſhall, therefore, confine our attention to the particular caſe in the queſtion. We have then $m = 2$, $n = \sqrt{(3)}$, and $rh = \frac{r^2}{2}$, therefore, having ſubſtituted, the fluxion to be integrated becomes

$$\frac{4\dot{s}}{r^2} = \frac{\dot{y}\sqrt{(1 + y^2)}}{\sqrt{(4y - y^2 - 1)}}.$$

where $y = 2 - \sqrt{(3)}.\,\text{coſ.}\,\varphi$, and, therefore, it is contained in the limits $2 + \sqrt{3}$, and $2 - \sqrt{3}$.

Let φ' denote the meaſure of another arch, reckoned from the oppoſite extremity of the diameter that φ is, and ſo related to φ that $(2 + \sqrt{(3)}.\,\text{coſ.}\,\varphi') \times (2 - \sqrt{(3)}.\,\text{coſ.}\,\varphi) = 1$: then $2 + \sqrt{(3)}.\,\text{coſ.}\,\varphi' = \frac{1}{y}$: alſo, let s' denote the conical ſurface, inſiſting on φ', then we ſhall obviouſly have

$$\frac{2\dot{s}'}{r^2} = \frac{\dot{y}\sqrt{(1 + y^2)}}{y^2\sqrt{(4y - y^2 - 1)}}.$$

Put $y + \frac{1}{y} = 2(u + 1)$: then $y = u + 1 + \sqrt{(u^2 + 2u)}$: alſo $\frac{1}{y} = u + 1 - \sqrt{(u^2 + 2u)}$: and $\frac{y^2 + 1}{4y - y^2 \mp 1} = \frac{u + 1}{1 - u}$:

therefore,

$$\frac{4\dot{s}}{r^2} = \frac{\dot{u}+u\dot{u}}{\sqrt{[1-u^2]}} + \frac{\dot{u}(u+1)^2}{\sqrt{[(1-u^2)(u^2+2u)]}}$$

$$\frac{4\dot{s}'}{r^2} = -\frac{\dot{u}+u\dot{u}}{\sqrt{[1-u^2]}} + \frac{\dot{u}(u+1)^2}{\sqrt{[(1-u^2)(u^2+2u)]}}$$

whence $\frac{2}{r^2}(\dot{s}+\dot{s}') = \frac{\dot{u}(u+1)^2}{\sqrt{[(1-u^2)(u^2+2u)]}}$. And $\frac{2}{r^2}(\dot{s}-\dot{s}') = \frac{\dot{u}+u\dot{u}}{\sqrt{[1-u^2]}}$. In theſe formulæ $u = \frac{(y-1)^2}{2y} = \frac{(1-\sqrt{[3]}\,\text{coſ.}\,\varphi)^2}{4-2\sqrt{[3]}\,\text{coſ.}\,\varphi}$; and, therefore, u is contained in the limits 1 and 0. The latter formula is evidently integrable, and we get $\frac{2}{r^2} \times (s-s') = -\frac{\pi}{2} + \psi - \text{coſ.}\,\psi$; where $\text{ſin.}\,\psi = u$, and $\pi = 3{\cdot}1415$ &c.; from which we learn, that the difference of the two conical ſurfaces s and s' is equal to a ſector of the baſe, together with a rectilineal ſpace: and it will be a maximum when $u = 0$, or when $\text{coſ.}\,\varphi = \frac{1}{\sqrt{[3]}}$ and $\text{coſ.}\,\varphi' = -\frac{1}{\sqrt{[3]}}$.

Let us now conſider the ſecond formula

$$\frac{2}{r^2}(\dot{s}+\dot{s}') = \frac{\dot{u}(u+1)^2}{\sqrt{[(1-u^2)(u^2+2u)]}} = \frac{\dot{u}(u+1)^2}{\sqrt{[(u^2+u)\cdot(2-u-u^2)]}}$$

Aſſume $u^2+u = 2z^2$: then $u = \frac{1}{2}\left\{\sqrt{[8z^2+1]}-1\right\}$:

And $\dot{u} = \frac{4z\dot{z}}{\sqrt{[8z^2+1]}}$: whence by ſubſtitution

$$\frac{2}{r^2}(\dot{s}+\dot{s}') = \frac{-\dot{z}(4z^2+1)}{\sqrt{[(1-z^2)(8z^2+1)]}} - \frac{\dot{z}}{\sqrt{[1-z^2]}}$$

The fluent of the latter part is obviously nothing more than the arch of which z is the ſine: Let $\text{coſ.}\,\psi = z = \sqrt{\left[\frac{u^2+u}{2}\right]} = \frac{1-y}{2y}\cdot\sqrt{\left[\frac{1+y^2}{2}\right]}$, and while y increaſes from $2-\sqrt{3}$ to $2+\sqrt{3}$, z will decreaſe from $+1$, to -1: then, integrating,

$$\frac{2}{r^2}(s+s') = \int\frac{-\dot{z}(4z^2+1)}{\sqrt{[(1-z^2)(8z^2+1)]}} + \psi'$$

The whole is now *reduced to* the integration of one formula, *which I ſhall call* $\dot{H}$, *and* which is comprehended in the forms that Mr. Landen has integrated by means of elliptic peripheries. Aſſume $\sqrt{[8z^2 + 1]} = x + \sqrt{[3 + x^2]}$ then $\frac{4z\sqrt{[1 - z^2]}}{\sqrt{[8z^2 + 1]}} = \sqrt{[1 - x^2]}$ and $\frac{8z\dot{z}}{(8z^2 + 1)} = \frac{\dot{x}}{\sqrt{[3 + x^2]}}$

therefore, $\frac{\dot{z}}{\sqrt{[(1 - z^2)(8z^2 + 1)]}} = \frac{\frac{1}{2}\dot{x}}{\sqrt{[(1 - x^2)(3 + x^2)]}}$.

But, $\dot{H} = \frac{2}{6} \cdot \frac{-\dot{z}}{\sqrt{[1 - z^2]}} \sqrt{[8z^2 + 1]} + \frac{1}{6} \cdot \frac{-\dot{z}}{\sqrt{[1 - z^2]}} \times \left\{ \sqrt{[8z^2 + 1]} + \frac{3}{\sqrt{[8z^2 + 1]}} \right\} = \frac{2}{6} \cdot \frac{-\dot{z}}{\sqrt{[1 - z^2]}} \sqrt{[8z^2 + 1]} + \frac{1}{6} \cdot \frac{-\dot{z}}{\sqrt{[1 - z^2]}} \times 2\sqrt{[3 + x^2]} = \frac{2}{6} \cdot \frac{-\dot{z}}{\sqrt{[1 - z^2]}} \sqrt{[8z^2 + 1]} + \frac{1}{6} \cdot \frac{-\dot{x}}{\sqrt{[1 - x^2]}} \sqrt{[3 + x^2]} + \frac{1}{6} \cdot \frac{-x\dot{x}}{\sqrt{[1 - x^2]}}$ where the fluents of the two firſt parts are obviously elliptic arches, and the third part is an algebraic quantity. Let $x =$ coſ. ψ''

$H = \int \dot{\psi}' \sqrt{[1 - \frac{8}{9}\,\text{ſin.}^2\,\psi'} + \frac{1}{3}\int \dot{\psi}'' \sqrt{[1 - \frac{1}{4}\,\text{ſin.}^2\,\psi''} + \frac{\text{ſin.}\,\psi''}{6}$ where the two firſt parts are arches of two ellipſes of which the excentricities are $\frac{2\sqrt{2}}{3}$ and $\frac{1}{2}$, the ſemi-tranſverſe of each being 1: the arches ψ' and ψ'' are reckoned from the extremity of the conjugate axis, and are defined by ordinates parallel to the tranſverſe equal to ſin. ψ' and ſin. ψ'' reſpectively.

The preceding analyſis enables us to aſſign a portion of the conic ſurface ſtanding on an indefinite arch of the baſe, as we have computed both the ſum and difference of the parts denoted by s and s'. But we are required only to find the whole ſurface of the cone which is now readily done, and is equal to

$$\frac{1}{4}\left\{E + \frac{2}{3}E'\right\} + \frac{\pi}{2}$$

where E denotes the periphery of the ellipſe of which the excentricity is $\frac{2\sqrt{[2]}}{3}$; E′ that of the ellipſe of which the excentricity is $\frac{1}{2}$; and $\pi = 3{\cdot}1416$.

Note. It may be proper to remark that the arch ψ'' becomes *a ſemicircle when* ψ' becomes a quadrant.

XVII.

XVII. QUESTION 17, *by* GEOMETRICUS.

A triangle being given by poſition, and two points being alſo given, it is required to draw lines through theſe points to interſect each other in one of the ſides of the triangle and meet the other ſides ſo as to cut off ſegments adjacent to given points in them, having to each other a given ratio.

FIRST SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

ANALYSIS. Fig. 25, Pl. 1. Let ABC be the triangle given by poſition, P, Q the given points, from which lines are to be drawn to meet in BC, one of its ſides, and F, G the given points in AB, AC the remaining ſides. Suppoſe the problem reſolved, that is, ſuppoſe a point D is found in BC, ſuch, that DP, DQ being drawn and produced to meet the given lines AB, AC in H and K reſpectively, the ratio of FH to GK may be given. Take GL to FB in the given ratio of GK to FH, thus L will be a given point, and the ratio of HB to KL the ſame with the given ratio of HF to KG. Draw PM parallel to AB meeting BDC in M; the line PM will evidently be given by poſition, therefore, M will be a given point, and a line drawn from Q to M will be given by poſition. Draw BR parallel to MQ, meeting DQ in N, then DR being drawn through a given point parallel to a line given by poſition, will alſo be given by poſition.

The triangles DMQ, DBN are ſimilar, hence BN : MQ :: BD : DM, but the triangles DMP, DBH, being alſo ſimilar BD : DM :: BH : MP, therefore, BN : MQ :: BH : MP, and alternately, BN : BH :: MQ : MP; now, MQ, MP are given lines, for the points M, P, Q are given, therefore, the ratio of BN to BH is given, but by hypotheſis and from what has been already ſhewn, the ratio of BH to LK is given. Thus it appears, that B and L are given points in BR, LA lines given by poſition, and that QKN is a line drawn from Q a given point without them, cutting off ſegments BN, LK adjacent to the given points B, L, and having to each other a given ratio; now this is a problem well known, and which has been conſtructed in various ways by writers on Geometry, for example by Simpſon, in his Elements of Geometry (Prob. 37, page 242, ſecond edit.) but more eſpecially by Dr. Halley, in his Tranſlation of the *Sectio Rationis* of Apollonius.

The manner of conſtruction and demonſtration follow ſo obviously from the preceding analyſis, that, for the ſake of brevity, we omit them.

SECOND SOLUTION, *by Mr.* JOHN BUTTERWORTH.

Let LMN (fig. 26, pl. 1.) be the given triangle, D and C the given points in the ſides MN and LN, and A and B the other given points. Suppoſe EH and EG to be drawn through the points A and B to meet the ſides LN and MN in H and G ſo that CH may have to DG the given ratio. From A draw AS parallel to LN meeting LM in S, join BS, and draw BA to meet LN at I and CA to meet LM at P, and join PB; on BS take BT = IC, and draw TK parallel to AB to meet BP in K; alſo KR parallel to SB to meet MN at R. Now, it is manifeſt, from Prop. IV. of Profeſſor Playfair's Paper on Priſms *, that if any two lines whatever, as AE, BE, be inflected from the points A and B to the line LM and meet KR in F and LC in H, KF will be equal to HC. Hence the problem is reduced to this, viz. Through the given point B to draw a right line GF to meet MN and KR at G and F, ſo that DG and KF may have a given ratio; the method of doing this is pointed out at page 25, vol. III. of the Old Series of the Repoſitory.

XVIII. QUESTION 18, *by* GEOMETRICUS.

Let A, B, be two given points in AB a ſtraight line given by poſition, and let C, D be two given points without that line; let CV, DV, be drawn meeting AB in F and G, ſo that AF may have to BG a given ratio: It is required to determine the *Locus* of the point V.

FIRST SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

ANALYSIS. Divide AB at H (fig. 27, 28, 29, 30, pl. 2.) ſo that AH may have to HB the given ratio of AF to BG, then H

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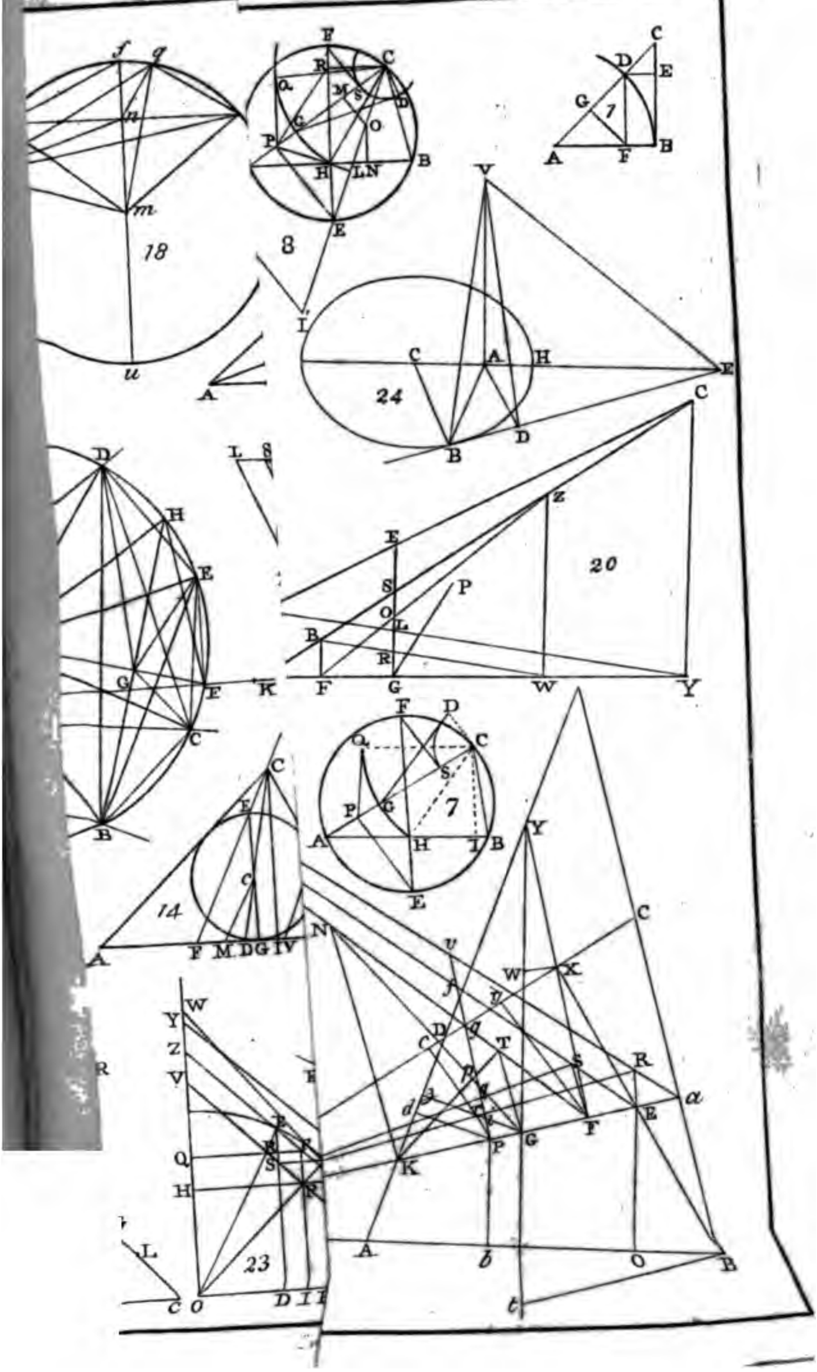
* *This Paper is printed in the Firſt Part of the Mathematical Tracts and Selections.*

Plate I. Fig. 1. to 26.



will be a given point, and FH will have to HG the given ratio that AF has to BG, or that AH has to HG. Now there will be ſeveral caſes of the problem, depending upon the poſition of the given points C, D with reſpect to the line AB and to the point H.

CASE I. Suppoſe that the line which joins the points C, D is parallel to AB (fig. 27.); join VH meeting CD in K, then becauſe FH : HG :: CK : KD, the ratio of CK to KD is given; therefore, the point K, and the line KH are given by poſition, ſo that in this caſe the locus of V is a ſtraight line given by poſition.

CASE II. Suppoſe that the line which joins the points C and D paſſes through H (fig. 28.); draw FK parallel to DG meeting HD in K, then HF : FG :: HK : KD, thus the ratio of HK to KD is given, and, therefore, K is a given point. Again, CK : KD :: CF : FV, but CK and KD are given magnitudes, therefore, the ratio of CF to FV is given, now C is a given point, and F is in a ſtraight line AB given by poſition, therefore, the locus of V in this caſe is alſo a ſtraight line given by poſition, which will be parallel to the given line AB.

CASE III. Next ſuppoſe that the line which joins the points C, D meets AB in E, (fig. 29.) ſo that the given ratio of FH to HG is the ſame as that of CE to DE, and ſo that HF, HG lie in the ſame, or in a contrary direction, according as CE, DE lie in the ſame, or in a contrary direction. Draw CM parallel to HG meeting DV in N. The ratio of CN to HG is compounded of the ratios of CN to FG and of FG to HG; but CN : FG :: VN : VG and by hypotheſis FG : HG :: CD : DE :: ND : DG, therefore, the ratio of CN to HG is compounded of the ratios of VN to VG, and of ND to DG, hence CN : HG :: VN × ND : VG × DG, thus it appears, that the locus of V is a parabola which paſſes through three given points H, C, D, and has its axis parallel to HG a line given by poſition, therefore, the parabola itſelf is given by poſition.

CASE IV. In the laſt place, let us ſuppoſe that the point H and the line which joins the given points C, D have any other poſitions than thoſe already conſidered. Join HK (fig. 30.) and draw FK parallel to DG meeting DH in K: then DH : HK :: GH : HF, that is in a given ratio, now D and H are given points, therefore, K is alſo a given point. Join CK, this line will be given by poſition, and will meet AB in a given point L, for otherwiſe CK muſt be parallel to AB, and, therefore, producing DC to meet AB in E, CE : ED :: KH : HD :: FH : HG, which is contrary to the hypotheſis. Draw HM, VQ parallel to CL, meeting FV, FE in N and Q, the line HM will evidently be given by poſition.

The triangles QFV, LFC are ſimilar, alſo the triangles QGV, LFK, hence

QF : QV :: FL : LC
and QV : QG :: KL : LF
therefore, ex æq. QF : QG :: KL : LC

Now KL, LC are given magnitudes, therefore, the ratio of QF : QG, and hence the ratio of QF to FG is given, but by hypotheſis the ratio of GF to FH is given; therefore, the ratio of QF to FH, or that which is equal to it, the ratio of VF to FN is given, ſo that the ratio of FV to NV is alſo given. We have now found that from the given point C there is drawn a line CV to meet two lines HM, HA given by poſition, ſo that FV, NV the diſtances of V from the lines given by poſition have to each other a given ratio, hence the locus of V is a given hyperbola as is pretty generally known, and as was perhaps firſt ſhewn by Dr. Barrow, in his Geometrical Lectures (Lect. VI.)

SECOND SOLUTION, *by Mr.* J. I.

Divide AB in E (fig. 31, pl. 2.) ſo that AE : EB :: AF : AG. Join CE and make CE : EH :: AF : FG or AE : EB. Draw DLH and CK parallel to it. Then, becauſe AE : EB :: AF : GB; therefore, AE : EB :: FE : EG :: KE : EL; conſequently, KF : GL :: KE : EL, or as CK to HL. Draw CR parallel to AB and divide DR in N, ſo that DN : NR :: DL : LH; draw NQSP parallel to AB meeting DV in Q, CV in P, and KC in S. Alſo divide KL in M, ſo that KM : ML :: DL : LH. And draw VO, MT parallel to DH. Then, from what has been ſhewn,

HL : LG :: CK : KF :: CS : SP,
but LD : LH :: DN : NR or CS;
therefore, LD : LG :: DN : PS :: DN : NQ;
therefore, PS = NQ.
Again, PS : CS or NR :: PO : OV,
and, DN : NQ or PS :: VO : OQ;
therefore, DN : NR :: PO : OQ :: DL : LH :: PM : ML.

But PS = NQ; therefore, PQ = NS = KL; therefore, PO = KM, OQ = ML, and, OT = PS = NQ.

Becauſe now OT = PS, and OP = TS; therefore, VO : PO or ST :: CS : SP or OT, and VO × OT = CS × ST. Conſequently the locus required is an hyperbola having TM and TN for its aſymptotes and paſſing through the point C.

The conſtruction is manifeſt from the Analyſis. It may be remarked that the locus is a ſtraight line when C and D are in a ſtraight line parallel to AB. The locus is alſo a ſtraight line when the three points C, E, D are in a ſtraight line. And when the line DH is parallel to AB the locus is no longer a hyperbola but a parabola.

THIRD SOLUTION, *by Mr.* WILLIAM SMITH, *Liverpool.*

CONSTRUCTION. Divide the given line AB (fig. 32, pl. 2.) in E, ſo that AE may be to EB in the ratio, which, by the queſtion AF is required to have to BG, and divide CD in e in the ſame ratio; join Ee and draw CA′, DB′ parallel thereto, meeting AB in A′ and B′. Join DE, which produce to L, ſo that DE : DL :: DB′ : A′C; through L draw an indefinite line parallel to Ee; draw CZ parallel to DL, meeting the parallel to Ee, through L, in Z; from D, on the oppoſite ſide of D to L, ſet off DM = LE, and join CM meeting LZ in H. Produce CZ to P′, ſo that the rectangle ZP′·ZL may be equal to the rectangle DL·LE; take the point P ſo that ZP : ZP′ :: HL : LM and complete the parallelogram ZHUP; find the point g in CA′ ſuch that the rectangle Cg·CH may be to ZP·ZH :: CH : CZ, and from g draw gO parallel to CH meeting PU in O; then with the centre O and aſymptotes gO, OP deſcribe an hyperbola whoſe power may be equal to the rectangle gO·gC, and it will be the locus required.

DEMONSTRATION. From D and C to any point V in the hyperbola draw CV (meeting AB in F, DL in f, HL in n, PU in i, and Ee in R) and DV (meeting AB in G, Ec in r, and HL in S), join SR and LV meeting SR in I and from S draw SK parallel to CV meeting DL in K; draw VQdW parallel to DL (meeting PO in Q, HL in d, and CA in W) and Vbo parallel to CH meeting PO in b and CA′ in o, and through n draw plqN (meeting CA′ in p, PO in t, a line through V parallel to PO in q and CM in N). Then, by conſtruction, bg·gO = Ob·bV, that is, by parallels, &c. og : ob :: Cg : bV, hence, bg is parallel to CV, and, therefore, bi (= Cg) a given line. By the ſimilar triangles CHn, Vib, Hn bV (= CH·ib) is given = CH·bg; therefore, ſince bV : tq :: CH : CZ, and alſo, by conſtruction, (CH·Cg) Hn·bV : HZ·ZP :: CH : CH, it follows that Hn·tq = HZ·ZP, that is, the figure Utqk = ZHUP, add to both HntV, and Zn·nt = Hn·dV(nq); but by conſtruction, nt (ZP) : ZP′ :: Hn : nN, therefore, Zn·ZP = Nn·dV, but by parallels Zn : nN :: ZL : Mf, therefore, ZL·ZP′ = Mf·dV, *alſo by conſtruction,* ZL·ZP′ = DL·LE, therefore, DL·LE = *Mf·dV, but by parallels,* DL : LK :: VD : SK (Vc), therefore, $DK \cdot dV = DL \cdot fK$. Whence, ſince $DD \cdot LE = Mf \cdot dV$ and $DL \cdot fK = DK \cdot DV$, $LE\,(MD) : fK :: Mf : DK$, or, dividendo et invertendo, $MD : Df :: fK : Df$, therefore, $MD\ (= EL) = fK$; but ſince from the points S, R, two parallels interſect Dm in f, K, and make the intercepted parts EL, fK equal to each other; therefore, the line joining the points S, R is parallel to Dm, therefore, $ER = SL$, whence, $Er : ER\,(SL) :: DE : DL ::$ (by conſtr.) $DB' : CA'$, that is, $DB' : Er :: A'C : ER$, or $B'G : GE :: A'F : FE$, or $B'G : A'F :: B'E : EA' :: BE : EA$; that is, BG is to AF in the ratio required.

Q. E. D.

Mr. Marrat *ſent an algebraical ſolution to this queſtion.*

XIX. QUESTION 19, *by Mr.* WILLIAM WALLACE, *of the Royal Military College, Great Marlow, Bucks.*

Let AB be the diameter of a circle, and FG a ſtraight line perpendicular to the diameter, meeting it in C, draw any two lines CHD, CKE meeting the circle in H, D, alſo in K, E; join DK, EH, meeting the line perpendicular to the diameter in F and G, then FC is equal to CG: Required the demonſtration?

FIRST SOLUTION, *by Mr.* THOMAS BAZLEY, *Bolton.*

CASE I. When the point C is within the circle. Fig. 33, Pl. 2.

Let the points where FG meet the circle be M, N; through G draw PGQ parallel to KD meeting KE in Q and DH in P. Then by the ſimilar triangles QEP, HPG, $QG : EG :: HG : PG$; therefore, $GE \times GH = GP \times GQ$; and by the ſimilar triangles QGC, KFC, we have, $KF : QG :: CF : CG$, and, $FD : GP :: CF : CG$. Hence, $KF \times FD : QG \times GB = GE \times GH :: CF^2 : CG^2$.

But by the property of the circle,

$GE \times GH = MG \times GN = CN^2 - CG^2$,
and $KF \times FD = MF \times FN = CM^2 - CF^2$;

conſequently, $CM^2 - CF^2 : CN^2 - CG^2 :: CF^2 : CG^2$,
and by compoſition, $CM^2 : CN^2 :: CF^2 : GG^2$.

But $CM = CN$, becauſe BC is perpendicular to MN; therefore, $CF = CG$, which was to be proved.

CASE II. When the point C is without the circle. Fig. 34, Pl. 2.

In the very ſame manner as in the firſt Caſe we ſhall have, $KF \times FD : GE \times GH :: CF^2 : CG^2$; now, through the centre O draw OG and OF, meeting the circle in I and L. Then by the property of the circle,
$KF \times FD = OF^2 - OL^2$, and $GE \times GH = OG^2 - OI^2$; therefore,
$OF^2 - OL^2 : OG^2 - OI^2 :: CF^2 : CG^2$, and by diviſion,
$OF^2 - OL^2 - CF^2 : OG^2 - OI^2 - CG^2 :: CF^2 : CG^2$;
but $OF^2 - CF^2 = OC^2 = OG^2 - CG^2$; therefore,
$OC^2 - OL^2 : OC^2 - OI^2 :: CF^2 : CG^2$;
but the two firſt terms of this proportion are evidently equal; and, therefore, $CF^2 = CG^2$, or $CF = CG$.

W. W. D.

SECOND SOLUTION, *by Mr.* WALLACE, *the Propoſer.*

This queſtion, owing to an accidental circumſtance, was propoſed in another place. I ſhall here give it a more general form, and ſhew that the ſame property belongs to any conic ſection whatever.

THEOREM. Let C (Fig. 35, Pl. 2.) be any point in the diameter of a conic ſection; MN a ſtraight line paſſing through C parallel to a tangent at the extremity of the diameter, CHD, CKE, any two ſtraight lines paſſing through C and meeting the curve in the points D, H, and E, K, join EH, DK, meeting MN in G and F, then CG is equal to CF.

Suppoſe the point C to be without the conic ſection, draw CP, CQ tangents to the curve at P and Q, and join PQ which will be parallel to MN, draw tangents to the curve at the points D, H, and E, K, let R and S be the interſections of the tangents, and L the interſection of the lines DK, EH, the points R, L, S, are in a ſtraight line; (See Robertſon's Treatiſe on Conic Sections, Book IV. Prop. 18.) but the points R, S, are in the line PQ produced (Robertſon, Book IV. Prop. 17.), therefore, the point L is in the line PQ.

Let CD meet PQ in T, then CD is harmonically divided at the points C, H, T, D, (Robertſon, Book IV. Prop. 16.) or DT : DC :: TH : HC, but becauſe TL is parallel to GF, DT : DC :: TL : CF, alſo HT : HC :: TL : CG, therefore, TL : CF :: TL : CG, and CF = CG.

When C is within the curve, then L will fall without it, and it will readily appear, from the prop. already quoted, that if tangents be drawn at the points in which MN cuts the curve, the point L will be in a ſtraight line paſſing through their interſection parallel to MN. If DC be produced to meet the parallel in T, the remainder of the demonſtration will be as before.

Solutions were likewiſe given by Meſſrs. Butterworth, Collins, Johnſon, Morley, Pickering, *and* Wright.

XX. QUESTION 20, *by Mr.* WILLIAM SMITH, *Liverpool.*

Given the foci of an ellipſis to deſcribe it ſo as to touch a circle given in poſition and magnitude.

SOLUTION, *by Mr.* THOMAS HOPPER.

Suppoſe the problem ſolved, and from the given foci conceive right lines drawn to the point where the circle and ellipſis are in contact; then, it is manifeſt that a tangent to the ellipſis at this point will be perpendicular to a right line drawn from it to the centre of the circle; which right line or radius will, therefore, biſect the angle formed by the two focal lines.

Hence the Problem is reduced to that of drawing, from two given points, two right lines to meet each other in the periphery of a given circle, ſo that if another right line be drawn from the point of concourſe to the centre of the circle, it ſhall make equal angles with the lines firſt drawn. Which has been done, *vide* page 223. of the Appendix to the 2d. Edit. of Simſon's Conic Sections; *vide* alſo, Examp. 7, Book 10, De l'Hoſpital's Conics.

And thus the Queſtion was anſwered by Mr. Wright.

XXI.

XXI. QUESTION 21, *by Mr.* JOHN LOWRY.

Having given any number of points A, B, C, D, &c. in the periphery of a given circle it is required to find another point P (in the periphery) ſuch that joining PA, PB, PC, &c. the ſum of the lines PA, PB, PC, &c. may be equal to a given line.

FIRST SOLUTION, *by Mr.* WM. SMITH, *Liverpool.*

ANALYSIS. Fig. 36, Pl. 2. Suppoſe the thing done and that P is the required point; join PA, PB, PC, PD, &c. Produce AP ſo that PX may be equal to PB, and join BX, then, the angle AXB being equal to half the angle APB is given; therefore, the locus of the point X is a circle given in poſition deſcribed on AB; let XQ drawn parallel to CP meet the circle ABX in Q; then, ſince the angle AXQ (=APC) is given, Q is a given point; from C draw CL parallel to AX meeting QX in L, then, the angle CLQ (= APC is given; therefore, the locus of the point L is a circle deſcribed on CQ and given in poſition. Produce PX to Y ſo that XY = XL = CP, and join YL meeting the circle CQL in R, then the angle QLR (= $\frac{1}{2}$AXL = $\frac{1}{2}$APC) is given, and R a given point; alſo, the angle AYR (= $\frac{1}{2}$AXL = $\frac{1}{2}$APC) is given, and the locus of the point Y is a circle given in poſition, paſſing through the points A, R, Y, and let YS drawn parallel to PD meet the circle AYR in S, then the angle AYS (= APD) is given, and S a given point; draw DM parallel to AY meeting SY in M, then the angle DMS (= APD) is given, and the locus of M a circle given in poſition and deſcribed on DS; take YZ (= YM) = DP, and join ZM meeting the circle DMS in T, then the angle SMT (= $\frac{1}{2}$APD) is given, and T a given point; alſo, the angle AZT (= $\frac{1}{2}$AYM = $\frac{1}{2}$APD) is given, and the locus of the point Z a circle given in poſition deſcribed on AT; but (*ſuppoſing the number of points four, as it is evident the reaſoning employed in the above Analysis, will apply, whatever the number of points may be*) ſince AZ (=AP + BP + CP + DP) is a given line, the locus of Z is alſo a circle deſcribed from the centre A, and interſection of this with the former locus ATZ will determine the point Z. Whence this

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CONSTRUCTION. On AB deſcribe a ſegment of a circle to contain an angle equal to half the angle APB, and apply the chord AQ in that ſegment to ſubtend an angle equal to the angle APC ſubtended by AC in the given circle; on CQ deſcribe a ſegment of a circle to contain an angle equal to APC, in which ſegment find the point R ſuch that the angle ſubtended by QR in the circle CQ may be equal to half the angle APC; on AR deſcribe a ſegment of a circle to contain an angle equal to half the angle APC, equal to half the angle ſubtended by RQ in the circle CRQ, in this ſegment apply the chord AS, to ſubtend an angle equal to the angle APD, and join DS, upon which deſcribe a ſegment of a circle to contain an angle equal to the angle APD ſubtended by AD, in the given circle, in this ſegment find the point T, ſo that the angle ſubtended by TS may be equal to half the angle APD, let AT be joined, upon which let a ſegment of a circle be deſcribed to contain an angle equal to half the angle APD to which, from A, apply AZ equal to the given ſum, meeting the circle ABCD in P, join AP, BP, CP, DP, and the thing is done.

SECOND SOLUTION, *by Mr.* JOHN WRIGHT, *Norley.*

ANALYSIS. Fig. 37, Pl. 2. Suppoſe the thing done, and AP, BP, CP, &c. drawn. Biſect the arc AB in Q, and join PQ, AQ, and AB. Then it is well known that $AP+BP : PQ$ in the given ratio of AB to AQ; therefore, when the number of given points is two, and $AP + BP$ given, PQ is given; and conſequently the locus of the required point P in this caſe is a circle of which the centre is Q and radius QP.

Draw the chord CQ, and having determined the point R in the arc CBQ, ſo that the chords CR, QR may obtain the given ratio of AB to AQ, join CP; then, becauſe $CR : QR :: AB : AQ :: PA + PB : PQ$, the rectangle $(PA + PB)\cdot QR = PQ\cdot CR = PR\cdot CQ - PC\cdot QR$ (Emerſon's Geom. IV. 32.); whence $(PA + PB + PC) \times QR = PR \times CQ$, or $PA + PB + PC : PR :: CQ : QR$ a given ratio; hence, when the number of given points is three, and $PA + PB + PC$ given, PR is given, and the locus of P is a circle of which the centre is R and radius RP.

Draw the chord DR, and in the arch DRC let the point S be ſo taken, that chords DS, RS may obtain the given ratio of CQ to RQ, and join PS. Since $DS : RS :: CQ : RQ :: PA + PB + PC : PR$, the rectangle $(PA+PB+PC)\cdot RS = PR\cdot DS = PS\cdot DR - PD\cdot SR$; hence $(PA + PB + PC + PD)\cdot RS = PS\cdot DR$, PS·DR, or PA + PB + PC + PD : PS :: DR : RS a given ratio; therefore, when the number of given points is four, and AP + BP + CP + DP given, PS is given, and the locus of P is a circle whoſe centre is S and radius SP. And thus the ſolution may be extended to any number of points.

THIRD SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

Let the given arches included between the points A, B, C, D, &c. be denoted by 2α, 2β, 2γ, &c. put 2φ for the arch PA, r for the radius of the circle, and a for the given ſtraight line equal to the ſum of the chords PA, PB, PC, PD, &c. which chords being equal to $2\,\text{ſin.}\ \varphi$, $2\,\text{ſin.}\ (\alpha+\varphi)$, $2\,\text{ſin.}\ (\beta+\varphi)$, $2\,\text{ſin.}\ (\gamma+\varphi)$, &c. reſpectively, we have,

$$2\left\{\text{ſin.}\ \varphi + \text{ſin.}\ (\alpha+\varphi) + \text{ſin.}\ (\beta+\varphi) + \text{ſin.}\ (\gamma+\varphi) + \&\text{c.}\right\} = a,$$

or, ſince $r \times \text{ſin.}\ (\alpha+\varphi) = \text{ſin.}\ \alpha\ \text{coſ.}\ \varphi + \text{coſ.}\ \alpha\ \text{ſin.}\ \varphi$, &c.

$$\left.\begin{array}{l} \frac{2\,\text{coſ.}\ \varphi}{r}\,(\text{ſin.}\ \alpha + \text{ſin.}\ \beta + \text{ſin.}\ \gamma + \&\text{c.}) \\ + \frac{2\,\text{ſin.}\ \varphi}{r}\,(r + \text{coſ.}\ \alpha + \text{coſ.}\ \beta + \text{coſ.}\ \gamma + \&\text{c.}) \end{array}\right\} = a.$$

Put $2\,(\text{ſin.}\ \alpha + \text{ſin.}\ \beta + \text{ſin.}\ \gamma + \&\text{c.}) = s$,
and $2\,(r + \text{coſ.}\ \alpha + \text{coſ.}\ \beta + \text{coſ.}\ \gamma + \&\text{c.}) = c$;
then $s\ \text{coſ.}\ \varphi + c\ \text{ſin.}\ \varphi = ar$.

Take μ ſuch an arch of the given circle that $\text{tan.}\ \mu = \frac{rs}{c}$, thus μ will be given, and $s = c \times \frac{\text{ſin.}\ \mu}{\text{coſ.}\ \mu}$, hence, by ſubſtitution and reduction,

$$\text{ſin.}\ \mu\ \text{coſ.}\ \varphi + \text{coſ.}\ \mu\ \text{ſin.}\ \varphi = \frac{ar}{c} \times \text{coſ.}\ \mu = \frac{ar}{c} \times \frac{r^2}{\sqrt{(r^2 + \text{tan.}^2\ \mu)}}$$

$$= \frac{ar^2}{\sqrt{(c^2+s^2)}}.$$

That is, $\text{ſin.}\ (\mu+\varphi) = \frac{ar}{\sqrt{(c^2+s^2)}}$.

Hence the arch φ is determined, and ſince ſin. $(\mu + \varphi)$ cannot exceed r, therefore, the given line a muſt not exceed $\surd(c^2 + s^2)$.

Ingenious Solutions were alſo received from Meſſrs. Butterworth, *and* Johnſon.

XXII. QUESTION 22, *by* AMICUS.

It is well known that if a denote an arch of a circle, radius being unity, the tangent of that arch is expreſſed by the ſeries $a + \frac{a^3}{3} + \frac{2a^5}{15} +$ &c. but it is not ſo generally known what is the ſcale of relation of the numeral coefficients of the terms: It is therefore required to determine that ſcale.

FIRST SOLUTION, *by Mr.* J. I.

Let $\tau = \tan.\ x$: then, $\frac{\dot{\tau}}{\dot{x}} = \frac{1}{\text{coſ.}^2 x} = \frac{2}{1 + \text{coſ.}\ 2x}$: or $\frac{\dot{\tau}}{\dot{x}} \times \frac{1 + \text{coſ.}\ 2x}{2} = 1$. Now

$$\frac{1 + \text{coſ.}\ 2x}{2} = 1 - \frac{2}{1.2}.x^2 + \frac{2^3}{1.2.3.4}.x^4 - \frac{2^5}{1.2.3.4.5.6}.x^6 + \&c$$

And let τ be aſſumed $= x + \frac{A^{(3)}}{1.2} \cdot \frac{x^3}{3} + \frac{A^{(5)}}{1.2.3.4} \cdot \frac{x^5}{5} + \frac{A^{(7)}}{1.2.3.4.5.6} \cdot \frac{x^7}{7} +$ &c. in which the law of the denominators is obvious, and the ſymbols, $A^{(3)}$, $A^{(5)}$, $A^{(7)}$, &c. denote indeterminate quantities of which the law of derivation is to be aſcertained; the numbers joined to the ſymbols mark the term of the ſeries to which each belongs, by pointing out the exponent of x in that term; then ſubſtituting the foregoing ſeries in the equation inveſtigated above; and putting the coefficients of the ſeveral powers of $x = 0$, we ſhall readily find,

$$A^{(3)} = 2,$$

$$A^{(5)} = \frac{4.3}{1.2}.2A^{(3)} - 2^3,$$

$$A^{(7)} = \frac{6.5}{1.2}.2A^{(5)} - \frac{6.5.4.3}{1.2.3.4}.2^3A^{(3)} + 2^5$$

$$A^{(9)} = \frac{8.7}{1.2}.2A^{(7)} - \frac{8.7.6.5}{1.2.3.4}.2^3A^{(5)} + \frac{8.7.6.5.4.3}{1.2.3.4.5.6}.2^5A^{(3)} - 2^7$$

or in general

$$A^{(2n+1)} = \frac{2n.2n-1}{1.2}.2A^{(2n-1)} - \frac{2n.2n-1.2n-2.2n-3}{1.2.3.4}.2^3A^{(2n-3)}$$

$$+ \frac{2n.2n-1.2n-2.2n-3.2n-4.2n-5}{1.2.3.4.5.6}.2^5A^{(2n-5)} - \&c.$$

In this formula, conſiſting of terms alternately poſitive and negative, each term is formed by multiplying together the correſponding terms of three other ſeries, viz:

The ſeries of quantities $A^{(2n-1)}$, $A^{(2n-3)}$, &c. reckoned backward which precede the quantity ſought in the ſeries of coefficients:

The ſeries of the odd powers of 2; 2, 2^3, 2^5, &c. and the third ſeries is nothing more than the 3d, 5th, 7th, &c. coefficients in the binomial $(a + b)^{2n}$

The law of continuation of the ſeries for the tangent is therefore completely aſcertained: And the method here given has the advantage of bringing out the ſeries $A^{(3)}$, $A^{(5)}$, $A^{(7)}$, &c. in integral numbers.

SCHOLIUM.

In like manner may the law of continuation of the ſeries for the ſecant be determined. Let

$$\text{ſecant } x = 1 + \frac{A^{(2)}}{1.2}.x^2 + \frac{A^{(4)}}{1.2.3.4}.x^4 + \frac{A^{(6)}}{1.2.3.4.5.6}.x^6 + \&c.$$

then

$$A^{(2)} = 1$$

$$A^{(4)} = \frac{4.3}{1.2}.A^{(2)} - 1$$

$$A^{(6)} = \frac{6.5}{1.2}.A^{(4)} - \frac{6.5.4.3}{1.2.3.4}.A^{(2)} + 1.$$

And in general

$$A^{(2n)}=\frac{2n.2n-1}{1.2}.A^{(2n-2)}-\frac{2n.2n-1.2n-2.2n-3}{1.2.3.4}.A^{(2n-4)}$$

$$+\frac{2n.2n-1.2n-2.2n-3.2n-4.2n-5}{1.2.3.4.5}.A^{(2n-6)}-\text{\&c.}$$

If we denote by M the meridional parts in Mercator's chart correſponding to the arch x: then

$$M=\int\frac{\dot{x}}{\text{coſ.}\,x}=x+\frac{A^{(2)}}{1.2}.\frac{x^3}{3}+\frac{A^{(4)}}{1.2.3.4}.\frac{x^5}{5}+\text{\&c.}$$ where

$A^{(2)}, A^{(4)}, A^{(6)}$ &c. are as determined above.

SECOND SOLUTION, *communicated by* PHILALETHES CANTABRIGIENSIS.

Many years have elapſed ſince an under-graduate of *Trinity College*, communicated to me, and to ſome other friends, the following method of computing the tangent of an arch in terms of the arch itſelf; which method is much eaſier than that which was uſed by Dr. Halley for the purpoſe about one hundred years ago, and by Mr. Emerſon in his Trigonometry about forty years ago, (and by later writers,) viz. by dividing the sine by the coſine.

If the radius of a circle be called 1, and if t be put for the tangent of an arch denoted by z, then, $\frac{\dot{t}}{1+tt}=\dot{z}$, and $z=t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}$, &c. which equations have often been given by the writers on fluxions. Now, from the first of theſe equations we have $\frac{\dot{t}}{\dot{z}}=1+tt$; and, if we would expreſs the value of t in terms of z, we know from the ſecond that it muſt be in ſeries proceeding by the 1ſt, 3d, 5th, &c. powers of z. Let t be aſſumed $=Az+Bz^3+Cz^5+Dz^7+Ez^9+Fz^{11}+Gz^{13}+Hz^{15}$, &c. then, $\frac{\dot{t}}{\dot{z}}=A+3Bz^2+5Cz^4+7Dz^6+9Ez^8+11Fz^{10}+13Gz^{12}+15Hz^{14}$, &c. But, ſince $\frac{\dot{t}}{\dot{z}}=1+tt$, and $1+tt$ is $=$

1 +

$$\begin{aligned}1+AAz^2+2ABz^4+2ACz^6+2ADz^8+2AEz^{10}+2AFz^{12}+2AGz^{14},\&c.\\ +BBz^6+2BCz^8+2BDz^{10}+2BEz^{12}+2BFz^{14},\&c.\\ +CCz^{10}+2CDz^{12}+2CEz^{14},\&c.\\ +DDz^{14},\&c.\end{aligned}$$

we now have an equation for determining the values of the coefficients A, B, C, &c. viz.

$$A+3Bz^2+5Cz^4+7Dz^6+9Ez^8+11Fz^{10}+13Gz^{12}+15Hz^{14},\&c.=$$
$$\left.\begin{aligned}1+AAz^2+2ABz^4+2ACz^6+2ADz^8+2AEz^{10}+2AFz^{12}+2AGz^{14},\&c.\\ +BBz^6+2BCz^8+2BDz^{10}+2BEz^{12}+2BFz^{14},\&c.\\ +CCz^{10}+2CDz^{12}+2CEz^{14},\&c.\\ +DDz^{14},\&c.\end{aligned}\right\}$$

Hence we have $A=1,$

$$B=\frac{AA}{3},$$
$$C=\frac{2AB}{5},$$
$$D=\frac{2AC+BB}{7},$$
$$E=\frac{2AD+2BC}{9},$$
$$F=\frac{2AE+2BD+CC}{11},$$
$$G=\frac{2AF+2BE+2CD}{13},$$
$$H=\frac{2AG+2BF+2CE+DD}{15},$$
&c.

where the law of continuation is evident.

But, ſince A is evidently $=1$, $B=\frac{1}{3}$, $C=\frac{2}{15}$, and $D=\frac{17}{315}$, the above equations may be expreſſed in ſimpler terms, as here below:

$$A=1,$$
$$B=\frac{A}{3},$$
$$C=\frac{2B}{5},$$
$$D=\frac{2C+\frac{1}{3}B}{7},$$
$$E=\frac{2D+\frac{2}{3}C}{9},$$
$$F=\frac{2E+\frac{2}{3}D+\frac{2}{15}C}{11},$$

$$G = \frac{2F + \frac{2}{3}E + \frac{4}{15}D}{13},$$

$$H = \frac{2G + \frac{2}{3}F + \frac{4}{15}E + \frac{17}{315}D}{15},$$

&c. &c.

where the law of continuation is again very obvious.

COROLL. Since it appears above, that $1 + tt$, the ſquare of the ſecant of the arch z, is $= 1 + 3Bz^2 + 5Cz^4 + 7Dz^6 + 9Ez^8 + 11Fz^{10} + 13Gz^{12} + 15Hz^{14}$, &c. we have only to extract the ſquare root of this ſeries (of which the coefficients are now known), and we ſhall have the value of the ſecant. This may be done as follows:

Aſſume $\sqrt{(1 + tt)} = 1 + \beta z^2 + \gamma z^4 + \delta z^6 + \epsilon z^8 + \zeta z^{10} + \eta z^{12} + \theta z^{14}$, &c. then, by involution and ſubſtitution, we ſhall have

$$1 + 3Bz^2 + 5Cz^4 + 7Dz^6 + 9Ez^8 + 11Fz^{10} + 13Gz^{12} + 15Hz^{14}, \&c =$$

$$\left.\begin{array}{llllllll} 1 + 2\beta z^2 & + 2\gamma z^4 & + 2\delta z^6 & + 2\epsilon z^9 & + 2\zeta z^{10} & + 2\eta z^{12} & + 2\theta z^{14}, \&c. \\ & + \beta\beta z^1 & + 2\beta\gamma z^6 & + 2\beta\delta z^8 & + 2\beta\epsilon z^{10} & + 2\beta\zeta z^{12} & + 2\beta\eta z^{14}, \&c. \\ & & & + \gamma\gamma z^3 & + 2\gamma\delta z^{10} & + 2\gamma\epsilon z^{12} & + 2\gamma\zeta z^{14}, \&c. \\ & & & & & + \delta\delta z^{12} & + 2\delta\epsilon z^{14}, \&c. \end{array}\right\}$$

from which equation,

$$\beta = \frac{3B}{2},$$

$$\gamma = \frac{5C^2 - \beta\beta}{2}$$

$$\delta = \frac{7D - 2\beta\gamma}{2},$$

$$\epsilon = \frac{9E - 2\beta\delta - \gamma\gamma}{2},$$

$$\zeta = \frac{11F - 2\beta\epsilon - 2\gamma\delta}{2},$$

$$\eta = \frac{13G - 2\beta\zeta - 2\gamma\epsilon - \delta\delta}{2},$$

$$\theta = \frac{15H - 2\beta\eta - 2\gamma\zeta - 2\delta\epsilon}{2},$$

&c. &c.

Here alſo the law of continuation is evident; and the expreſſions may be ſimplified by writing for β, γ, &c. their numerical values as faſt as they ariſe.

XXIII. QUESTION 23, *by Mr.* JAMES CUNLIFFE, *Bolton.*

To find the leaſt biquadrate number, that can be divided into four integral parts, ſuch, that the ſum of every two of them may be a ſquare number.

SOLUTION, *by Mr.* J. I.

Let n^4 denote the biquadratic number ſought, and a, b, c, d, its four integral parts; then the queſtion requires

$$\begin{aligned} a+b&=\mu^2\\ a+c&=\mu'^2\\ a+d&=\mu''^2\\ c+d&=\nu^2\\ b+d&=\nu'^2\\ b+c&=\nu''^2 \end{aligned}$$

whence we get

$$n^4=a+b+c+d=\mu^2+\nu^2=\mu'^2+\nu'^2=\mu''^2+\nu''^2.$$

We ſee, therefore, that the biquadratic number ſought muſt be reſolvable into three ſquares three different ways; and, this condition being fulfilled, we derive

$$\begin{aligned} a&=\tfrac{1}{2}(\mu^2+\mu'^2-\nu''^2)\\ b&=\tfrac{1}{2}(\mu^2+\nu''^2-\mu'^2)\\ c&=\tfrac{1}{2}(\mu'^2+\nu''^2-\mu^2)\\ d&=n^4-\tfrac{1}{2}(\mu^2+\mu'^2+\nu''^2), \end{aligned}$$

from which formulas we infer that the three ſquares, μ^2, μ'^2, ν''^2, muſt either be all even, or two of them muſt be odd and only one even, in order that no fractions may ariſe from the diviſions by 2; and likewiſe that the ſum of every two of them muſt be greater than the third, and at the ſame time the ſum of all the three leſs than $2n^4$, in order to avoid negative numbers.

It is, then, neceſſary to find a biquadratic number that may be divided into two ſquares three different ways. We owe to Fermat the curious diſcovery that every prime number exceeding a multiple of four by unity, or comprehended in the form $4x+1$, is the ſum of two ſquares. And this property, which is univerſal for prime numbers of the form $4x+1$, belongs excluſively *to that form ; ſo that* no prime number, which is not of the form $4x + 1$, can poſſibly be the ſum of two ſquares. This curious propoſition was firſt demonſtrated by the celebrated Euler; and to effect the demonſtration it was neceſſary to prove this other propoſition, viz. That every diviſor of a number which is the ſum of two ſquares that are prime to one another, is likewiſe the ſum of two ſquares. From theſe principles we eaſily gather that the biquadratic number n^4, we are ſeeking, muſt have prime diviſors of the form $4x + 1$; for if it had none it could not poſſibly be the ſum of two ſquares. And as we want a leaſt value of n^4, we may ſet aſide all other diviſors; for if n^4 had any other diviſors, theſe could only enter, as common meaſures, into every two ſquares that compoſe n^4. We may, therefore, ſuppoſe n^4, and conſequently n, to have no diviſors but prime numbers of the form $4x + 1$.

And in the firſt place let us ſee whether a prime number p of the form $4x + 1$ will anſwer for n. We have already mentioned that p is the ſum of two ſquares $\alpha^2 + \beta^2$; and we have now to add that p can be the ſum of two ſquares only one way.

In like manner we ſhall find $p^2 = (\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2$; nor can p^2 be reſolved into two ſquares in another way. Proceeding to the fourth power, we ſhall find,

$$p^4 = (\alpha^4 - 6\alpha^2\beta^2 + \beta^4)^2 + (4\alpha^3\beta - 4\alpha\beta^3)^2$$
$$p^4 = p^2\alpha^2 + p^2\beta^2.$$

And theſe two are all the ways that p^4 can be reſolved into two ſquares. Therefore, no ſingle prime number of the form $4x + 1$ will anſwer the conditions required.

Let us now try the product of two prime numbers of the ſame form as p and p'. Then according to what has been ſhewn we ſhall have

$$p^4 = \left(\alpha^4 - 6\alpha^2\beta^2 + \beta^4\right)^2 + \left(4\alpha^3\beta - 4\alpha\beta^3\right)^2$$
$$p^4 = p^2\alpha^2 + p^2\beta^2.$$

And in like manner,

$$p'^4 = \left(\alpha'^4 - 6\alpha'^2\beta'^2 + \beta'^4\right)^2 + \left(4\alpha'^3\beta' - 4\alpha'\beta'^3\right)^2$$
$$p'^4 = p'^2\alpha'^2 + p'^2\beta'^2.$$

Therefore, combining the two values of p^4 with the two values of p'^4, the biquadratic $p^4p'^4$ will be the product of the ſums of two ſquares four different ways. But the product of two ſums of two ſquares is the ſum of two ſquares twice:

$$\text{for } (M^2 + N^2)\cdot(P^2 + Q^2)$$
$$= (MP + NQ)^2 + (MQ - NP)^2$$
$$= (MP - NQ)^2 + (MQ + NP)^2.$$

Therefore, by combining the two values of p^4 with the two values of p'^4, the biquadratic p'^4p^4 will be the ſum of two ſquares no leſs than eight different ways. To theſe eight we may add four other ways, by combining p^4 with the two values of p'^4, and p'^4 with the two values of p^4. Therefore, the fourth power of a number which is the product of two primes of the form $4x + 1$, is the ſum of two ſquares, no leſs than twelve different ways.

We ſee then that no prime number of the form $4x + 1$, can ſatisfy the conditions required in the queſtion. But a number which is the product of two ſuch primes is the ſum of two ſquares twelve different ways, one ſquare of each ſum being even, and one odd; therefore, as every combination of three out of the twelve, will give one ſolution of the queſtion: and, as twelve things admit of 220 combinations of three, there will be ſo many different ſolutions, unleſs ſome are excluded as producing negative numbers. It is eaſy to ſee that one combination of three will give only one integral ſolution.

The two leaſt primes of the form $4x + 1$ are 5 and 13, and their product 65 is conſequently the leaſt number that will ſatisfy the queſtion. We have, $5 = 2^2 + 1$, and $13 = 3^2 + 2^2$: whence

$$65^4 = 3713^2 + 2016^2$$
$$65^4 = 2047^2 + 3696^2$$
$$65^4 = 2145^2 + 3640^2$$

which are only three of the twelve ways that 65^4 is the ſum of two ſquares: And taking the three even ſquares for μ, μ', and ν'', our formulæ give $a = 1826720$; $b = 2237536$; $c = 11422880$, and $d = 2363489$, which is one ſolution.

In like manner if we take another three of the twelve ways that 65^4 is the ſum of two ſquares, as

$$65^4 = 2047^2 + 3696^2$$
$$65^4 = 2145^2 + 3640^2$$
$$65^4 = 4095^2 + 1040^2,$$

we ſhall derive another ſolution; $a = 746208$; $b = 12914208$; $c = 335392$; $d = 3854817$.

XXIV. QUESTION 24, *by* JACK CHANCE.

A in $a + b$ relations, tells b lies; and B in $c + d$ relations tells d of them falſe: What is the probability of the truth of a ſtory in which *they both* agree?

SOLUTION,

SOLUTION, *by the Propoſer.*

$(a+b)\times(c+d) = ac+bc+ad+bd =$ all the caſes.

$ac =$ the number of times both agree in telling truth,

$bd =$ the number of times both agree in telling lies.

$bc =$ the times B tells truth, and A lies.

$ad =$ the times A tells truth, and B lies.

Therefore, $ac+bd =$ the whole number of times they both agree in telling both truth and lies.

But they are ſuppoſed to *agree* in their relation (by the queſtion):

therefore, $\frac{ac}{ac+bd} =$ the probability required.

COROL. If the ſtory were *untold*, the probability of their both telling true would be $\frac{ac}{ac+bc+ad+bd}$, that of their both lying would be $\frac{bd}{ac+bc+ad+bd}$, and that of their contradicting each other $\frac{bc+ad}{ac+bc+ad+bd}$.

Meſſrs. Bazley, Codling, Hearding, *and* Hopper, *alſo ſent Solutions to this queſtion.*

XXV. QUESTION 25, *by* QUINBUS FLESTRIN.

Given the baſe, the difference of the adjacent angles, and a line from the vertical angle to the middle of the baſe to conſtruct the plane triangle.

FIRST SOLUTION, *by Mr.* JOHN WRIGHT, *Norley.*

CONSTRUCTION. Biſect the given baſe AB in O, (fig. 38, *pl. 2.*) with the right line ON, forming the angle NOB equal *to the complement of half* the given difference; and draw BQ perpendicular to ON. Conſtitute the right-angled triangle ONC, having its hypothenuſe OC equal the given biſecting line, and ſuch, that its area may be equal to that of the triangle OQB; then draw AC, BC, and ABC is the triangle required.

DEMONSTRATION. Becauſe OQ × BQ = ON × NC, if the right line CI, terminating in AB, be drawn parallel to ON, it will biſect the angle ACB, as appears from Prob. 36, page 306 of Simpſon's Algebra. Hence the difference of the angles ABC and BAC is equal to twice the complement of CIB or NOB, the given magnitude by conſtruction.

Q. E. D.

SECOND SOLUTION, *by Mr.* JOHN DAWES, *Birmingham.*

ANALYSIS. Let ABC (fig. 39, pl. 2.) be the required triangle. Circumſcribe it with the circle AMCB, and draw CM parallel to the baſe AB; continue CE to G, unite M, G; and M, E; and becauſe EG is given (Euc. III. 35.); and ME and the angle MGC = the difference of the angles at the baſe are given by the queſtion; therefore, the triangle MGE is given; conſequently the triangle ABC is given. Hence this

CONSTRUCTION. Upon the give line CE (as radius) continued, and centre E deſcribe the ſemicircle NMC and take EG ſo that GEC may be equal to the ſquare of half the baſe; make the angle EGM equal half the given difference of the angles at the baſe, circumſcribe the triangle CGM with the circle GMCB, and parallel to MC through E, draw AEB; then AC, CB being joined, ACB will be the triangle required.

Solutions were alſo received from Meſſrs. Butterworth, Collins, Farey, Johnſon, Hopper, Morley, Myers, Pickering, *and* Scurr.

XXVI. QUESTION 26, *by Mr.* JAMES CUNLIFFE.

In a plane triangle there is given the perimeter, one of the angles, and the rectangle of the ſides including that angle, to conſtruct it.

FIRST

FIRST SOLUTION, *by Mr.* JOHN WRIGHT, *Norley.*

CONSTRUCTION. Fig. 40, pl. 2. Draw AB = the given perimeter, and make the angle OAB = ABO = half the given one; with the centre O and radius OA or OB deſcribe a circle, cutting BO produced in D; divide DB in Q, ſo that BQ may have to DQ the ratio which the rectangle of the ſides has to the ſquare of BO, and concentric to the circle DAB, deſcribe another through Q, cutting AB in I, through which point draw the diameter COIH, and having joined AH, BH, make the angles AHE, BHF, reſpectively equal to BAH, ABH, ſo ſhall HEF be the triangle required.

DEMONSTRATION. The angle HEF + HFE = 2HAB + 2HBA = HOB + HOA = AOB; conſequently, EHF = 2OAB = the given quantity; and EH being = EA, and FH = FB, EH + FH + EF = AB, the given perimeter. Moreover becauſe the angle HEI = BOI, the triangles EHI, BOI, are equiangular, as alſo are the triangles FHI, AOI, for a ſimilar reaſon.

Hence $EH \cdot HF : AO \cdot BO :: HI^2 : IB \cdot IA \, (HI \cdot IC) :: HI : IC$; that is, $EH \cdot HF : BO^2 :: BQ : DQ$.

Q. E. D.

SCHOLIUM. If the circle QII falls wholly below AB, the thing is impoſſible.

SECOND SOLUTION, *by Mr.* BAZLEY.

ANALYSIS. Suppoſe ACB (fig. 41, pl. 2.) the triangle required, C the given angle; then ſince the rectangle of the ſides AC × CB is given as well as the angle C, the area of the triangle is given; but the area of the triangle is equal to half the rectangle under the perimeter and the radius of the inſcribed circle, and ſince the perimeter is given, the radius of the inſcribed circle is alſo given. Now, if O be the centre of the inſcribed circle, and COL be drawn to meet the circumſcribing circle in L, and OP be let fall perpendicular to CB; we ſhall have the right-angled triangle COP wholly given, becauſe the angle OCP is equal to half the given one, and OP being the radius of the inſcribed circle; therefore, CO is given. But the rectangle AC × CB is given, and is *well known* to be equal to OC × (2LO + OC); but OC is *given, conſequently*, LO and LC are given. Whence this

Construction. Make ACB equal to the given angle, which biſect with the given line CL, on L as a centre, and with the radius LO deſcribe a circle cutting CA, CB in A and B; join AB, and ACB is the required triangle, as is plain from the Analyſis, and page 7, Vol. II. of the Old Series of the Repoſitory.

And thus the Queſtion was anſwered by Meſſrs. Butterworth, Cavill, *and* Farey.

Third Solution, *by Mr.* J. Collins, *Kenſington.*

Analysis. Let ACB (fig. 42, pl. 2.) be the required triangle, ACB the given angle which biſect with the indefinite right line CFW. In CA produced take CP = half the given perimeter, make PW perpendicular to AC, and let it meet CW in W; then the right-angled triangle CPW is wholly given, and CS is known, becauſe $CS \times (CF + SF) = CS \times SW = AC \times CB$ the given rectangle; therefore, all the parts of the triangle are known, and the conſtruction evident.

Thus the Queſtion was anſwered by Meſſrs. Dawes *and* Hopper.

Fourth Solution, *by Mr.* Morley, *London.*

Draw any two lines CP, CP′ (fig. 42, pl. 2.) to contain the given angle, and take CP, CP′ each equal to half the perimeter, and erect the perpendiculars PW, P′W meeting each other at W, and join CW which is known to biſect the vertical angle ACB. In CW find the point S, ſo that $CS \times SW =$ the given rectangle, biſect SW in F and with the radius FS and centre F deſcribe the circle ASBW, cutting AC, CB in A and B; then AB being joined, ACB will be the triangle required.

Mr. Johnſon *alſo anſwered this Queſtion.*

XXVII. QUESTION 27, *by Mr.* THOMAS BAZLEY, *Bolton.*

In any plane triangle whereof C is the vertex, O the centre of the inſcribed circle, Q the centre of the circle touching the baſe and ſides produced, D the point where the line biſecting the vertical angle cuts the baſe, and S the point where the ſaid line cuts the circumſcribing circle below the baſe. Then the points being joined it is QD : CD : : SD : OD; required the demonſtration.

FIRST SOLUTION, *by Mr.* JOHN FAREY.

Let the triangle ABC (fig. 43, pl. 2.) be circumſcribed by the circle ACBS, let its vertical angle C be biſected by the line CQ, cutting the baſe in D, and the circumſcribing circle in S; on S as a centre deſcribe a circle to paſs through A, and cut CQ in O and Q. Then, by Mr. *Lowry's* 1ſt. Prop. *of the Repoſitory, Vol.* II. *page* 7, *Old Series,* O is the centre of the inſcribed circle *above* the baſe, and Q the centre of the inſcribed circle *below* the baſe; the circumference alſo paſſes through B the other extremity of the baſe. Now, ſince the circles ACBS and AOBQ cut each other, and ADB joins their points of interſection; we have,

by Euc. III. 35. $CD \times SD = AD \times BD$, alſo $OD \times QD = AD \times BD$;

Hence $CD \times SD = OD \times QD$, or $QD : CD :: SD : OD$.

Q. E. D.

SECOND SOLUTION, *by Mr.* JOHN BUTTERWORTH.

Let ABC (fig. 44, pl. 2.) be any plane triangle circumſcribed by the circle ACBS; join the points as directed in the queſtion; join alſo the points A, S; draw OE to make the angle OEC equal to the angle ODB. Then it is well known that the points C, O, D, S, Q, are in the ſame ſtraight line CQ, and that $QS = SA = SO$; and the angle OEC being equal to the angle ODB, that is equal to the angle ADS, and *the angle DCB* equal to the angle SAD, the triangles COE and SAD are ſimilar; hence, AS (QS) : SD :: CO : OE, and by compoſition QS + SD = QD : SD :: CO + OE : OE, but OE is equal to OD, becauſe the angles OEC and ODB are equal; therefore, QD : SD :: CD : OD, or QD : CD :: SD : OD.

Q. E. D.

THIRD SOLUTION, *by Mr.* JOHN DAWES.

Let ABC be the triangle (fig. 45, pl. 2.). Becauſe C, O, D, S, Q, are in a right line, perpendiculars from thoſe points to the baſe AB will form ſimilar triangles.

By hypotheſis QD : CD :: SD : OD, theref. HD : DE :: DK : DL; but, DK = DL + LK and HD = HE + ED = EL + ED; therefore, ED+EL : ED::DL+LK : DL, or EL : ED :: LK:LD; therefore, ED : EL :: LD : LK, which is a well known property.

FOURTH SOLUTION, *by Mr.* THOS. HOPPER, *Mancheſter.*

Let ABC (fig. 45, pl. 2.) be the triangle circumſcribed by a circle whoſe diameter SF is perpendicular to the baſe AB; C, O, D, S, and Q, the points mentioned in the queſtion. Join SB and from Q upon AB, and CA produced let fall the perpendiculars QH, QI, which (as is well known) are equal to each other; alſo SB = SO and the angle SBE = QCI. Wherefore by ſimilar triangles,

IQ : IE :: CQ (= CD + QD) : BS (= SD + OD),
and HQ = IQ : SE :: QD : SD;
conſequently, CD + QD : SD+OD :: QD : SD, or
CD + QD : QD :: SD + OD : SD; and by diviſion
CD : QD :: OD : SD; whence, QD : CD :: SD : OD.

Q. E. D.

According to one or other of theſe methods were the Solutions given by Meſſrs. Bazley, Cavill, Collins, Johnſon, *and* Morley.

XXVIII. QUESTION 28, *by Mr.* BAZLEY.

Given the vertical angle, the radius of the inſcribed circle, and the ſum, difference, or ratio of the rectangle of the two ſides including the vertical angle and the rectangle of the two lines drawn from the extremes of the baſe to the centre of the inſcribed circle; to conſtruct the plane triangle.

SOLUTION, *by Mr.* JOHN CAVILL, *Beighton.*

ANALYSIS. Suppoſe ACB (fig. 46, pl. 2.) the triangle required, O the centre of the inſcribed circle through which COS is drawn meeting the circumſcribing circle below the baſe in S; now it is well known that a circle deſcribed about the point S with the radius SA or SB will paſs through O and interſect the greater ſide AC in b, making $Cb = CB$. Let R and P be the points of contact of the radius of the inſcribed circle, with the baſe AB and the ſide CB; and join OR, OP. Then, becauſe the angle $PCO =$ half the given vertical angle, and the angle P right, and ſince OP the radius of the inſcribed circle is given, the hypothenuſe OC is given. Again, by a well known theorem $2SO \times OR = AO \times OB$, and by what is done above, $AC \times CB = AC \times Cb =$ (by Euc. Cor. to 36. III.) $OC \times (2SO + OC)$. Hence, when the ſum or difference of the rectangles is given, $(2SO + OC) \times OC \pm 2SO \times OR$ is given; that is, $2SO \times OC + OC^2 \pm 2SO \times OR = 2SO \times (OC \pm OR) + OC^2$ is given, but ſince OR and OC are given, SO is given.

Laſtly, when the ratio of the rectangles is given we ſhall have (by taking $AC \times CB : AO \times OB :: n : m$ in the given ratio) $n : m :: 2SO \times OC + OC^2 : 2SO \times OR$; therefore, ſince CO and OR are given, SO is given. Wherefore, SO being found, all the caſes of the conſtruction may be as follows.

CONSTRUCTION. Having drawn the lines CA, CB to include the given angle and having alſo drawn the given inſcribed circle to touch them in P and p; through O the centre draw COS making OS of the given length found in the Analyſis, and on S as a centre with the radius SO deſcribe the circle AOB cutting AC and CB in A and B reſpectively; join AB and ABC will be the triangle required.

And thus the Queſtion was anſwered by Meſſrs. Bazley, Butterworth, Collins, Dawes, Johnſon, *and* Morley.

XXIX. QUESTION 29, *by Mr.* BAZLEY.

There is an hemiſpherical bowl, whoſe diameter at the brim is 20 inches, placed with its axis perpendicular to the horizon; an uniformly heavy thin inflexible rod or right line, whoſe length is 24 inches and weight 8 ounces, is ſuſtained in equilibrio by reſting on the edge of the brim at ſome diſtance from the upper extremity, whilſt the lower end reſts againſt the concave ſurface of the bowl. It is required to determine by a general inveſtigation, the poſition of the rod and the reſpective preſſures at the two points of ſupport.

FIRST SOLUTION, *by* AMICUS.

Let ARE (fig. 47, pl. 2.) be a ſection of the bowl and rod perpendicular to the horizon; O the centre of the bowl, and AOE an horizontal diameter; let G be the middle of the rod DR, or its centre of gravity, and draw GH perpendicular to AE and OK perpendicular to DR.

Put $OE = r = 10$; $RG = a = 12$; and the part of the rod within the bowl, $RE = x$; then, $EK = \frac{x}{2}$, and $OK = \sqrt{(r^2 - \frac{x^2}{4})}$: but $OE : OK :: EG : GH$, that is, $r : \sqrt{(r^2 - \frac{x^2}{4})} :: x - a : GH = \frac{1}{r}(x-a)\sqrt{(r^2 - \frac{x^2}{4})}$.

In whatever point of view the Theory of Statics has been conſidered, the following propoſition has either been aſſumed as a ſelf-evident maxim, or it has been ſhewn to be a neceſſary conſequence of the firſt principles on which every Theory is built; viz. That a ſyſtem of heavy bodies will be in equilibrium when the common centre of gravity has reached the point of loweſt deſcent, or of higheſt aſcent; or to ſpeak more mathematically, when the common centre of gravity is in that particular point of the locus which it deſcribes, when it paſſes from deſcending to aſcending, or aſcending to deſcending. But although the ſyſtem of bodies will be equally in equilibrium when the common centre of gravity exactly occupies the point of loweſt deſcent, or of higheſt aſcent, yet there will be a great difference in the ſtates of equilibrium in the two caſes. For in the one caſe, when the centre of gravity is in the point of loweſt deſcent, or when it paſſes from deſcending to aſcending, the equilibrium will have ſtability; and although the centre of gravity be diſplaced a little from the exact ſituation it ſhould occupy, it will have a tendency to regain its place, and will oſcillate about the point of equilibrium. On the other hand, when the centre of gravity is in the point of higheſt aſcent, or when it paſſes from aſcending to deſcending, the equilibrium will be unſtable; and if the ſyſtem be moved ſo as to diſplace the centre of gravity from its preciſe place ever ſo little, there will be no tendency to regain the ſituation of equilibrium; but, on the contrary, the ſyſtem will remove more and more from it.

In the preſent problem it is manifeſt that the centre of gravity G will deſcend, and the ſituation of the rod, where G is the loweſt, will be the equilibrium required, Now G will be in the loweſt place, when GH is a maximum, that is, when its fluxion, making x variable, is $= 0$. Therefore, $(r^2 - \frac{x^2}{4}) - \frac{x}{4}(x-a) = 0$; whence, $x^2 - \frac{a}{2}x = 2r^2$, and $x = \sqrt{(2r^2 + \frac{a^2}{16})} + \frac{a}{4}$.

In the particular caſe propoſed $x = \sqrt{209} + 3 = 17{\cdot}45$, &c.

To determine the preſſures at E and R draw RL perpendicular to AE, and let it meet OK produced in S. Then the forces by which the rod is ſuſtained in equilibrium are theſe, viz. the gravity of the rod acting in the direction SR; the preſſure on the brim of the bowl at E acting perpendicularly to the rod, or in the direction KS; and laſtly, the preſſure at R acting in the direction RO, perpendicularly to the bottom of the bowl: and the proportions of the forces will be the ſame as the proportions of the ſides of the triangle SOR formed by lines parallel to directions in which the forces act; or, the forces will be as the ſines of the angles of the triangle SOR, that is, as the lines RK, KO, OL. Hence, putting w for the weight of the rod, the preſſure on the brim of the bowl $= \frac{OK}{RK} \times w = \frac{\sqrt{(r^2 - \frac{x^2}{4})}}{\frac{1}{2}x} \times w = 4{\cdot}47$oz.

And the preſſure on the bottom $= \frac{OL}{RK} \times w = (\frac{x}{r} - \frac{2r}{x}) \times w = 4{\cdot}8$ ounces.

SECOND SOLUTION, *by the Propoſer.*

Let ABCOD (fig. 48, pl. 2.) be a ſection of the bowl and rod paſſing through the axis CB, and the rod ROGD, G the middle of the rod, or its centre of gravity; from O draw OS perpendicular to RD, and from D through the centre C draw DCS to meet OS in S, from S draw SIG, meeting ACO in I; join AD, and draw GT parallel to DS, meeting SO in T. Then by the nature of the Queſtion, the re-action of the edge of the bowl at O is in the direction OS, and the re-action of the concavity at D is in the direction DCS; therefore, by Emerſon's Mechanics, prop. 53. 4to Edit. the line SIG paſſing through the rod's centre of gravity is perpendicular to the horizon; whence, SI is perpendicular to ACO. This being premiſed, put AO the diameter of the bowl $= d$, GD or GR $= a$, and DO $= x$. Then, AD $=$ SO $= \sqrt{(d^2 - x^2)}$, and by the ſimilar triangles ADO, GOS, it is OD : AD :: AD $=$ OS : OG, that is, $x : \sqrt{(d^2 - x^2)} :: \sqrt{(d^2 - x^2)} : \frac{d^2 - x^2}{x} =$ GO, whence, DG $= a = x - \frac{d^2 - x^2}{x}$; therefore, $2x^2 - d^2 = ax$, or $x^2 - \frac{1}{2}ax = \frac{1}{2}d^2$, and conſequently, $x = \sqrt{(\frac{1}{2}d^2 + \frac{1}{16}a^2)} + \frac{1}{4}a$, by which the poſition of the rod is determined, and in the caſe propoſed, where $d = 20$, $a = 12$, we get $x =$ OD $=$ 17·45683 inches.

Again, by the Principles of Mechanics, the rod is ſuſtained by three forces, whoſe quantities and directions are as SG, GT, TS, reſpectively; that is, the weight of the rod, the re-action of the concavity at D, and the re-action of the brim at O. Hence putting the weight of the rod $= w$, by the ſimilar triangles SGT, SDG, we get

SG : GT :: SD : SG :: $d : \frac{d}{x}\sqrt{(d^2 - x^2)} :: w : \frac{\sqrt{(d^2 - x^2)}}{x} \times w$ $=$ the preſſure in the direction SD $=$ (in the preſent caſe where w 8 oz. and $x = 17{\cdot}45683$) 4·472784 ounces.

Alſo, SG : ST :: SD : DG :: $d : a :: w : \frac{a}{d} \times w =$ the preſſure in the direction SO $=$ 4·8 oz. in the preſent caſe.

Corollary. If $a = d$, the expreſſion for x becomes $= d$; and in this caſe the expreſſion for the preſſure againſt the concavity becomes $= 0$, whilſt that for the preſſure on the brim becomes $= w$. Again, when a vaniſhes as is the caſe when a weight $= w$ is connected to the lower extremity of the rod, (the rod itſelf being conſidered to have no weight) we have $x = \frac{1}{2}d^2$, which ſhews that the point D then coincides with B the loweſt point in the bowl; and in this caſe the expreſſion for the preſſure againſt the concavity becomes $= w$, whilſt that for the brim vaniſhes, as it ought. Moreover it evidently appears that a muſt not exceed d, as otherwiſe the rod would fall out of the bowl.

THIRD SOLUTION, *by Mr.* H. CODLING, *South Walſham.*

Suppoſe BIH (fig. 49, pl. 2.) a perpendicular ſection of the bowl through the centre C, CI the axis, RD the rod in the required poſition, G its centre of gravity (or middle of the rod), RZ, LGM and BK parallel, and RK perpendicular to the axis CI. Then, in order to ſimplify the problem, let us ſuppoſe the weight of the rod to be collected in the point G, for ſuch ſuppoſition will make no difference with regard to the preſſures on R and B.

Now, if the weight of the rod be repreſented by BR, it is evident, by the reſolution of forces, that it will be ſuſtained by two forces RK and BK, the one acting perpendicular and the other parallel to the axis of the veſſel; then, if the total preſſure be repreſented by RK, that exerted at R will evidently be as RL, and that at B as the ſum of the forces BK, KL, or by compoſition as BL; but theſe forces cannot be known till the poſition of the rod is determined, which may be done as follows: It is a known principle in Mechanics, that the centre of gravity will deſcend to the loweſt place it can get when the rod reſts in equilibrio; therefore, MG muſt be a maximum; then making $HZ = v$, $BH = d$, $GR = r$, by ſim. △s, $HR^2 = HB \cdot HZ = dv$, and $RB = \surd(d^2 - dv)$, therefore, $GB = \surd(d^2 - dv) - r$, and $d : \surd(dv) :: \surd(d^2 - dv) - r : GM = \dfrac{\surd(d^3 v - d^2 v^2) - r\surd(dv)}{d} =$ a maximum;

this being put into fluxions and reduced gives $v = \frac{1}{2}d - \dfrac{r^2 + r\surd(r^2 + 8d^2)}{8d} = 4{\cdot}76295$; conſequently, $BL = HR = \surd(dv) = 9{\cdot}760072$, $RL = d - 2v = 10{\cdot}474099$, and $BR = 17{\cdot}45683$. Now putting $w =$ the weight of the rod $= 8$ oz. we have the preſſure on $R = \dfrac{BL}{RB} \times w = \surd\left(\dfrac{dv}{d^2 - dv}\right) \times w =$ $4{\cdot}47274$ ounces and the preſſure at $B = \dfrac{RL}{RB} \times w =$ $\dfrac{d - 2v}{\surd(d^2 - dv)} \times w = 4{\cdot}8$ ounces.

XXX.

XXX. PRIZE QUESTION 30, *by* PHILALETHES CANTABRIGIENSIS.

Given $\frac{6\dot{x}x^5}{x^6 - x^4 + 2x^3 + 5x^2 - 6x + 6} = \dot{y}$, in which equation x and y begin together to find the value of y when x becomes $= 10$.

FIRST SOLUTION, *by the Propoſer*, PHILALETHES CANTABRIGIENSIS.

The denominator of the given fraction is the product of the three trinomial factors, $xx + 3x + 3$, $xx - 2x + 2$, and $xx - x + 1$, which may be diſcovered by Sir Iſaac *Newton's* method of diviſors, or more eaſily by the method explained by Madam *Agneſi*, in her *Analytical Inſtitutions*, Vol. I. p. 160. We may therefore, reſolve the given fraction into three fractions, by the method explained in p. 138, of Vol. II. of the work before mentioned, thus: Aſſume $\frac{\dot{x}x^5}{x^6 - x^4 + 2x^3 + 5x^2 - 6x + 6}$ $= \frac{Ax\dot{x} + D\dot{x}}{xx + 3x + 3} + \frac{Bx\dot{x} + E\dot{x}}{xx - 2x + 2} + \frac{Cx\dot{x} + F\dot{x}}{xx - x + 1}$, (omitting the coefficient 6, in the numerator of the given fraction, which will a little ſhorten the calculation) and find the values of A, B, C, &c. which will be as follow: viz.

$$A = \frac{180}{7 \times 61}, \qquad D = \frac{459}{14 \times 61},$$

$$B = \frac{44}{61}, \qquad E = \frac{48}{61},$$

$$C = -\frac{1}{7}, \quad \text{and} \quad F = -\frac{3}{14}.$$

Now, the fraction

$$\frac{Ax\dot{x} + D\dot{x}}{xx + 3x + 3} \text{ is evidently} = \frac{Ax\dot{x} + \frac{3}{2}A\dot{x}}{xx + 3x + 3} + \frac{(D - \frac{3}{2}A)\dot{x}}{xx + 3x + 3},$$

$$\frac{Bx\dot{x} + E\dot{x}}{xx - 2x + 2} \;\ldots\ldots\ldots = \frac{Bx\dot{x} - B\dot{x}}{xx - 2x + 2} + \frac{(E + B)\dot{x}}{xx - 2x + 2},$$

$$\frac{Cx\dot{x} + F\dot{x}}{xx - x + 1} \;\ldots\ldots\ldots = \frac{Cx\dot{x} - \frac{1}{2}C\dot{x}}{xx - x + 1} + \frac{(F + \frac{1}{2}C)\dot{x}}{xx - x + 1};$$

the correct fluents of which, are,

$$\text{1ſt.}\quad \tfrac{1}{2}\,\text{A} \times \text{h. l.}\left(\frac{xx}{3} + x + 1\right),$$

$$\text{2d.}\ + \tfrac{1}{2}\,\text{B} \times \text{h. l.}\left(\frac{xx}{2} - x + 1\right),$$

$$\text{3d.}\ + \tfrac{1}{2}\,\text{C} \times \text{h. l.}\ (xx - x + 1),$$

$$\text{4th.}\ + \tfrac{4}{3}(\text{D} - \tfrac{3}{2}\,\text{A}) \times \left\{\begin{array}{l}\text{circ. arch of tang. } x + \frac{3}{2} \\ - \text{circ. arch of tang. } \frac{3}{2}\end{array}\right\} \text{and radius } \frac{\sqrt{3}}{2},$$

$$\text{5th.}\ + (\text{E} + \text{B}) \times \left\{\begin{array}{l}\text{circ. arch of tang. } x - 1 \\ + \text{circ. arch of tang. } 1\end{array}\right\} \text{and radius } 1,$$

$$\text{6th.}\ + \tfrac{4}{3}(\text{F} + \tfrac{1}{2}\,\text{C}) \times \left\{\begin{array}{l}\text{circ. arch of tang. } x - \frac{1}{2} \\ + \text{circ. arch of tang. } \frac{1}{2}\end{array}\right\} \text{and radius } \frac{\sqrt{3}}{2}.$$

And the values of theſe fluents, (according to my calculation,) when x becomes $= 10$, are as below : viz.

$$\begin{array}{rcl@{\qquad}rcl}
 & & + & & & - \\
\text{1ſt.} & = & 0{\cdot}799195, & \text{3d.} & = & 0{\cdot}322204, \\
\text{2d.} & = & 1{\cdot}339322, & \text{4th.} & = & 0{\cdot}049113, \\
\text{6th.} & = & 0{\cdot}330490. & \text{5th.} & = & 0{\cdot}147248; \\
\hline
+ & & 2{\cdot}469007 & - & & 0{\cdot}518565 \\
- & & 0{\cdot}518565 & & & \\
\hline
 & & 1{\cdot}950442 = \frac{y}{6}, & & & \\
 & & 6 & & & \\
\hline
\end{array}$$

and $y = 11{\cdot}702652$, which was required.

SECOND SOLUTION, *by Mr.* I. H. HEARDING, *Adderbury*.

It is eaſy to diſcover, by Sir Iſaac *Newton's* method of diviſors, or ſtill more eaſily by *Agneſi's Analytical Inſtitutions*, Vol. I. p. 160, that the denominator of the given fraction is the product of $(x^2 + 3x + 3) \times (x^2 - x + 1) \times (x^2 - 2x + 2)$; therefore, by the above-mentioned Inſtitutions, Vol. II. page

138,

138, we may aſſume $\frac{x^5}{x^6 - x^4 + 2x^3 + 5x^2 - 6x + 6} = \frac{Ax + P}{x^2 + 3x + 3} + \frac{Bx + Q}{x^2 - x + 1} + \frac{Cx + R}{x^2 - 2x + 2}$, (the quantity $6\dot{x}$ being here purpoſely omitted to ſhorten the calculation). Then, by reducing theſe fractions to a common denominator, and equating the coefficients of the homologous terms, we ſhall have $A + B + C - 1 = 0$, $-3A + B + 2C + P + Q + R = 0$, $5A - B + C - 2P + Q + 2R = 0$, $-4A + 5P - Q + R = 0$, $2A + 6B + 3C - 4P = 0$, and $2P + 6Q + 3R = 0$; and from theſe equations A is found $= \frac{180}{7 \cdot 61}$, $B = -\frac{1}{7}$, $C = \frac{44}{61}$, $P = \frac{459}{14 \cdot 61}$, $Q = \frac{3}{14}$, and $R = -\frac{48}{61}$. Now, by ſubſtituting $v - \frac{3}{2}$, $u + \frac{1}{2}$, and $w + 1$, reſpectively for x, in the three foregoing fractions, in order to exterminate the ſecond terms of the denominators, and then multiplying the firſt ſide of the reſulting equation by $\dot{x}$, the firſt fraction on the laſt ſide by $\dot{v}$, the ſecond by $\dot{u}$, and the third by $\dot{w}$, there will at length ariſe

$$\frac{\dot{x}x^5}{x^6 - x^4 + 2x^3 + 5x^2 - 6x + 6} = \frac{Av\dot{v} + S\dot{v}}{vv + \frac{3}{4}} + \frac{Bu\dot{u} + T\dot{u}}{uu + \frac{3}{4}} + \frac{Cw\dot{w} + V\dot{w}}{ww + 1},$$

where $-\frac{3}{2}A + P = -\frac{81}{14 \cdot 61}$ is put $= S$, $\frac{1}{2}B + Q = \frac{1}{7} = T$, and $C + R = -\frac{4}{61} = V$.

The three fractions on the right hand ſide of the laſt equation are evidently equal to the ſum of theſe ſix following, viz.

$$= \frac{Av\dot{v}}{vv + \frac{3}{4}} + \frac{S\dot{v}}{vv + \frac{3}{4}} + \frac{Bu\dot{u}}{uu + \frac{3}{4}} + \frac{T\dot{u}}{uu + \frac{3}{4}} + \frac{Cw\dot{w}}{ww + 1} + \frac{V\dot{w}}{ww + 1};$$

the fluents of whichgives the fluent of

$$\frac{\dot{x}x^5}{x^6 - x^4 + 2x^3 + 5x^2 - 6x + 6} =$$

$\frac{1}{2}A \times$ hyp. log. $(vv + \frac{3}{4}) + \frac{1}{2}B \times$ h. l. $(uu + \frac{3}{4}) + \frac{1}{2}C \times$ h. l. $(ww + 1)$

$+ \frac{4S}{3} \times$ circular arc to tangent v and radius $\frac{1}{2}\sqrt{3}$,

$+ \frac{4T}{3} \times$ circular arc to tangent u and radius $\frac{1}{2}\sqrt{3}$,

$+ V \times$ circular arc to tangent w and radius 1.

And, by reſtoring the values of v, u, and w, this laſt expreſſion becomes =

$$\tfrac{1}{2}\times \text{h.l.}\,(x^2+3x+3)+\tfrac{1}{2}\text{B}\times \text{h.l.}\,(x^2-x+1)+\tfrac{1}{2}\text{C}\times \text{h.l.}\,(x^2-2x+2)$$

$+\frac{4S}{3}\times$ circular arc to tangent $x+\frac{3}{2}$ and radius $\frac{1}{2}\sqrt{3}$,

$+\frac{4T}{3}\times$ circular arc to tangent $x-\frac{1}{2}$ and radius $\frac{1}{2}\sqrt{3}$,

$+\text{V}\times$ circular arc to tangent $x-1$ and radius 1.

But as this ſhould vaniſh when $x=0$, the fluent of $\frac{\dot{x}r^5}{x^6-x^4+2x^3+5x^2-6x+6}=\frac{y}{6}$ when properly corrected is

$$\tfrac{1}{2}\text{A}\times \text{h.l.}\,(\tfrac{1}{3}x^2+x+1)+\tfrac{1}{2}\text{B}\times \text{h.l.}(x^2-x+1)+\tfrac{1}{2}\text{C}\times \text{h.l.}(\tfrac{1}{2}x^2-x^2+1)$$

$$\left.\begin{array}{l}+\frac{4S}{3}\\ +\frac{4T}{3}\\ +\text{V}\end{array}\right\}\begin{array}{c}\times\text{circular}\\ \text{arc to}\\ \text{tangent}\end{array}\left\}\begin{array}{l}x+\frac{3}{2}\\ x-\frac{1}{2}\\ x-\frac{1}{2}\end{array}\right\{\begin{array}{c}\textit{minus}\text{ the}\\ \text{arc whoſe}\\ \text{tangent}=\end{array}\left\}\begin{array}{r}\frac{3}{2}\\ -\frac{1}{2}\\ -1\end{array}\right\{\begin{array}{c}\text{and}\\ \text{common}\\ \text{radius}=\end{array}\left\}\begin{array}{c}\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3}\\ 1.\end{array}\right.$$

$= 1{\cdot}9504408$

And, conſequently, $y = 1{\cdot}9504408\times 6 = 11{\cdot}7026448$.

This Queſtion was alſo anſwered by Meſſrs. Johnſon, Mabbott, *and* Wright.

PHILALETHES CANTABRIGIENSIS *is requeſted to ſend to Mr.* GLENDINNING's *for the Medal for Solving the Prize Queſtion.*

* * *

ERRATA.

In Queſtion 51, No. I. For, " is $\frac{1}{2}\sqrt{(h^2+r^2)}+\frac{1}{4}S+\frac{1}{4}s$:" Read, " is a mean proportional between $\frac{1}{2}\sqrt{(h^2+r^2)}+\frac{1}{4}S+\frac{1}{4}s$ *and* r."

NOTICES

RELATING TO MATHEMATICS.

I. REGULAR POLYGON OF SEVENTEEN SIDES.

Mr. GAUSS, of Brunſwick, publiſhed at Leipſic, in 1801, a work called *Diſquiſitiones Arithmeticæ.* In this work the author announces that we may always inſcribe a regular polygon of $2^n + 1$ ſides in a circle, when n is a whole number, and $2^n + 1$ a prime number. Mr. LEGENDRE has given to the French National Inſtitute a demonſtration of this very curious propoſition, in the caſe when the number of ſides is 17. It is founded on theſe two lemmas.

LEMMA I. Let a be an arch of a circle, m and n two whole numbers, then

$$2 \text{ coſ. } ma \text{ coſ. } na = \text{coſ. } (m - n)\, a + \text{coſ. } (m + n)\, a.$$

LEMMA II. Let a be the nth part of the whole circumference, n being a whole number, then

$$\text{coſ. } a + \text{coſ. } 2a + \text{coſ. } 3a + \text{\&c.} \ldots\ldots + \text{coſ. } na = 0.$$

Theſe two propoſitions are generally known.

By the ſecond of them we have

$$\text{coſ.} a + \text{coſ.} 2a + \text{coſ.} 3a + \text{\&c.} \ldots + \text{coſ.} 15a + \text{coſ.} 16a + \text{coſ.} 17a = 0.$$

and, becauſe $\text{coſ. } 17a = 1.$

$$\text{coſ.} a + \text{coſ.} 2a + \text{coſ.} 3a + \text{\&c.} \ldots + \text{coſ.} 15a + \text{coſ.} 16a = -1 \ldots (\text{E}).$$

Now, if we remark that $\text{coſ. } a = \text{coſ. } 16a$, $\text{coſ. } 2a = \text{coſ. } 15a$, $\text{coſ. } 3a = \text{coſ. } 14a$, &c. it will immediately appear that equation (E) may be decompoſed into two others (E′) and (E″).

$$\left.\begin{matrix} \text{coſ. } a + \text{coſ. } 3a + \text{coſ. } 5a + \text{coſ. } 7a \\ + \text{coſ. } 9a + \text{coſ. } 11a + \text{coſ. } 13a + \text{coſ. } 15a \end{matrix}\right\} = -\tfrac{1}{2} \ldots (\text{E}')$$

$$\left.\begin{matrix} \text{coſ. } 2a + \text{coſ. } 4a + \text{coſ. } 6a + \text{coſ. } 8a \\ + \text{coſ. } 10a + \text{coſ. } 12a + \text{coſ. } 14a + \text{coſ. } 16a \end{matrix}\right\} = -\tfrac{1}{2} \ldots (\text{E}'')$$

Again, equation (E′) may be reſolved into theſe two,

$$\left.\begin{array}{l}\text{coſ. } 3a + \text{coſ. } 5a + \text{coſ. } 7a + \text{coſ. } 11a = m \\ \text{coſ. } a + \text{coſ. } 9a + \text{coſ. } 13a + \text{coſ. } 15a = n\end{array}\right\} \cdots\cdots (e')$$

where m and n are two ſuch quantities that $m + n = -\frac{1}{2}$ and $mn = -1$.

This ſecond reſolution may be verified by taking the product of the equations, (e') and changing, by the firſt lemma, the products of the coſines of the arches into the coſines of their ſum and difference.

Again each of the equations (e') may be decompoſed into two others by making $m = p + q$, $n = r + s$, thus,

$$\left.\begin{array}{ll}\text{coſ. } 3a + \text{coſ. } 5a = p & \text{coſ. } a + \text{coſ. } 13a = r \\ \text{coſ. } 7a + \text{coſ. } 11a = q & \text{coſ. } 9a + \text{coſ. } 15a = s\end{array}\right\} \cdots\cdots (f)$$

Here, $pq = rs = \frac{1}{4}$; the truth of theſe equations may be verified as before.

We have now obtained ſix equations

$$\begin{array}{lll} m + n = -\frac{1}{2} & p + q = m & r + s = n \\ mn = -1 & pq = -\frac{1}{4} & rs = -\frac{1}{4}\end{array}$$

and from theſe the values of m, n, p, q, r, s, may be found by the reſolution of equations, each of the ſecond degree.

The four equations (f) may, by lemma 1, be written thus,

$$\begin{array}{ll} \text{coſ. } 3a + \text{coſ. } 5a = p & \text{coſ. } 6a + \text{coſ. } 7a = q \\ \text{coſ. } 3a \text{ coſ. } 5a = \frac{s}{2} & \text{coſ. } 6a \text{ coſ. } 7a = \frac{r}{2} \\ \text{coſ. } a + \text{coſ. } 4a = r & \text{coſ. } 2a + \text{coſ. } 9a = s \\ \text{coſ. } a \text{ coſ. } 4a = \frac{p}{2} & \text{coſ. } 2a \text{ coſ. } 9a = \frac{q}{2} \end{array}$$

theſe eight coſines, viz. coſ. a, coſ. $2a$, coſ. $3a$, coſ. $4a$, coſ. $5a$, coſ. $6a$, coſ. $7a$, coſ. $9a$, may be found by the reſolution of equations of the ſecond degree. The remaining eight, viz. coſ. $16a$, coſ. $15a$, coſ. $14a$, coſ. $13a$, coſ. $12a$, coſ. $11a$, coſ. $10a$, coſ. $8a$, are reſpectively equal to the former eight, as has been already obſerved. Thus it appears that the deſcription of a regular polygon of 17 ſides is a problem which may be effected by means of ſtraight lines and circles; that is, by the methods of conſtruction employed in the elements of plane geometry.

II. ORBITS OF THE NEW PLANETS.

The elements of the orbits of the two new Planets have now been determined with tolerable accuracy: they are thus given by La Lande *.

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* *From the Journal de Phyſique*, Brumaire, an. 12.

Piazzi, or Ceres, diſcovered January 1, 1801.

Revolution 4 years, 7 months, 10 days.

Mean longitude, January 1, 1804	10*s*	11°	59′
Annual motion	2	18	14
Aphelion	10	26	44
Node	2	21	6
Equation of the orbit	0	9	3
Excentricity			0·097
Inclination	0	10	37

Olbers, or Pallas, diſcovered March 28, 1802.

Revolution 4 years, 7 months, 11 days.

Mean longitude, January 1, 1804	9*s*	29°	53′
Annual motion	2	18	11
Aphelion	10	1	7
Node	5	22	28
Equation of the orbit	0	28	25
Excentricity			0·2463
Inclination	0	34	39

III. FIGURE OF THE EARTH.

The length of a degree of the meridian, meaſured in Lapland by Maupertuis, le Monnier, &c. agreed ſo ill with theory, and with the other meaſurements, as to give good grounds to ſuſpect conſiderable error. The Swediſh Aſtronomer Melanderhielm has cauſed all the operations of that meaſurement to be carefully repeated: and the reſult of this labour differs from the old determination no leſs than 196 toiſes: the length of a degree of the meridian in latitude 66° 20′ coming out 57,209 toiſes inſtead of 57,405 according to the French Academicians. The new reſult agrees better with the various meaſurements in different latitudes.

The French Aſtronomer Mechain is employed in continuing to the Balearian Iſlands the meaſurement of the meridian in France, conducted by him and Delambre.

In the Eaſt Indies, Major Lambton, who is employed by the Preſidency of Madras to ſurvey the Myſore country, has already meaſured an arch of the meridian 1° 34′ 56″·43 in extent, which gives the degree in latitude 12° 32′ equal to 60,494 fathoms.

IV.

IV. MATHEMATICAL WORKS LATELY PUBLISHED.

The Philoſophical Tranſactions of the Royal Society of London. Abridged by Charles Hutton, L. L. D. F. R. S. George Shaw, M. D. F. R. S. F. L. S. Richard Pearſon, M. D. F. A. S.

A Courſe of Mathematics, deſigned for the uſe of the Officers and Cadets of the Royal Military College. Vol. I. By Iſaac Dalby, *Profeſſor of Mathematics in the ſaid College.*

Tracts on the Reſolution of Cubic and Biquadratic Equations. By Francis Maſeres, Eſq. F. R. S. *Curſitor Baron of the Exchequer.*

A Geometrical Treatiſe of Conic Sections, in Four Books, to which is added a Treatiſe on the Primary Properties of Conchoids, the Ciſſoid, the Quadratrix, Cycloids, the Logarithmic Curve, and the Logarithmic, Archimedian, and Hyperbolic Spirals. By the Rev. Abram Robertſon, A. M. F. R. S. *Savillian Profeſſor of Geometry in the Univerſity of Oxford.*

An Experimental Enquiry into the Nature and Propagation of Heat. By John Leſlie.

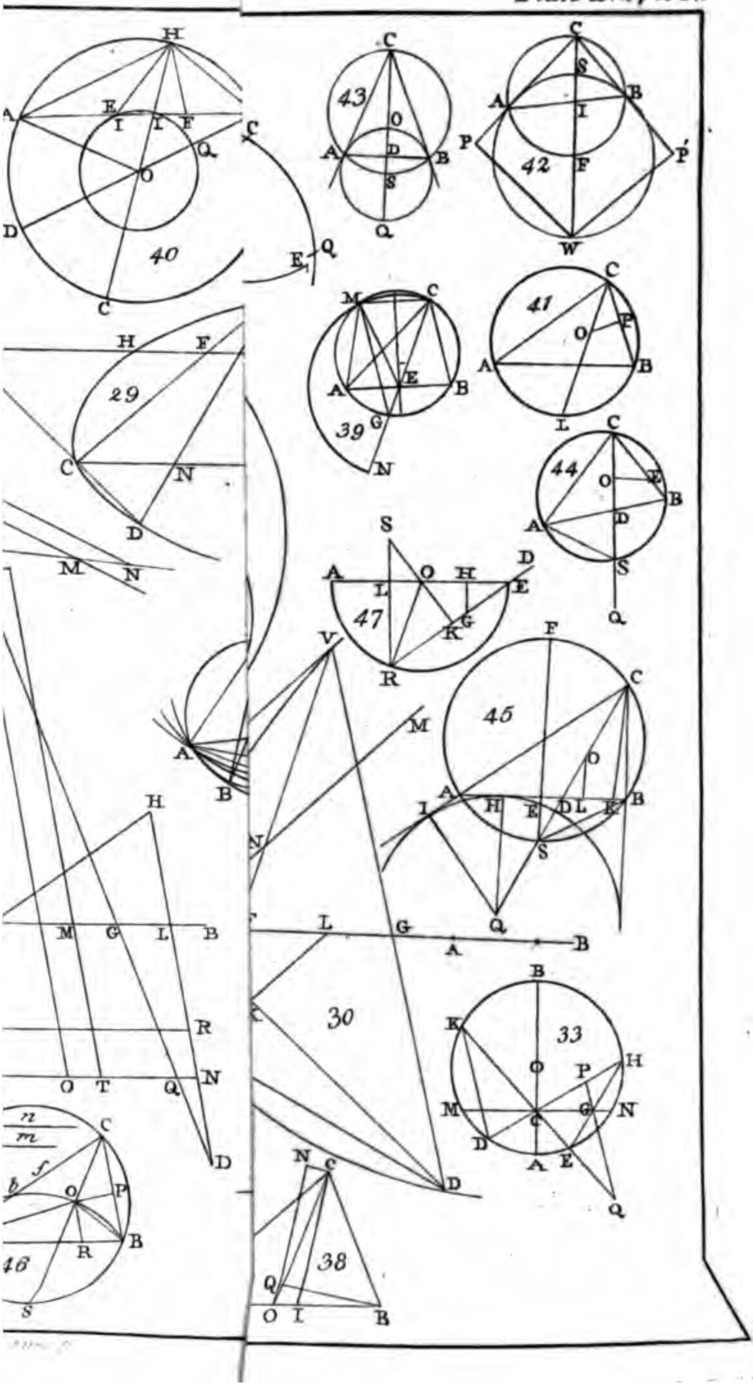
Recreations in Mathematics and Natural Philoſophy; containing Diſſertations concerning the moſt proper Subjects to excite curioſity and attention to Mathematical and Philoſophical Sciences. The whole treated in an eaſy manner, and adapted to the comprehenſion of all who are in the leaſt initiated in the Sciences of Arithmetic, Geography, Navigation, Architecture, Chemiſtry, &c. &c. Firſt compoſed by M. Ozanam, and now tranſlated into Engliſh, and improved, with many Additions. By Charles Hutton, L. L. D. F. R. S.

A General Hiſtory of Mathematics, from the earlieſt times to the middle of the Eighteenth Century. Tranſlated from the French of John Boſſut. *To which is affixed, a Chronological Table of the moſt eminent Mathematicians. By* John Bonnycaſtle.

MATHEMATICAL WORKS IN THE PRESS.

Profeſſor Robinſon, of Edinburgh, has in the Preſs a Work on the Elements of Mechanical Philoſophy, which will contain the ſubſtance of the excellent Lectures he has for many years read in the Univerſity of Edinburgh.

A New Edition of Profeſſor Playfair's *Elements of Geometry is ſoon to appear.*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ARTICLE II.

Solutions to Queſtions propoſed in Number I.

I. QUESTION 31, *by Mr.* T. MILNER.

Being out with a quadrant, ſome of my pupils were deſirous of knowing the height of the pariſh church ſpire, which riſes to a conſiderable height in the middle of the town of Sheffield; to gratify their curioſity I found two ſtations in Church Street, diſtant from each other 70 yards, exactly on a level with the bottom of the ſpire, ſuch that the altitude of the top of the ſpire at each ſtation was equal to 35¾ degrees; and again, in the middle of this ſtation line, the altitude of the ſpire's top was found to be 36½ degrees. From hence it is required to find geometrically the height of the ſaid ſpire, and its diſtance from the laſt ſtation, ſuppoſing the quadrant to be held 5 feet from the ground at each ſtation.

FIRST SOLUTION, *by Mr.* THOMAS HOPPER, *Mancheſter.*

CONSTRUCTION. On any given line AB (fig. 52, pl. 3.) taken at pleaſure, deſcribe the two right-angled triangles BCA, BDA, making the angle BCA = 36½° and BDA = 35¾°. Draw CF at right angles to AC and equal to 35 yards, and thereto apply AE = AD, let AE produced meet FG, drawn parallel to AC, in G; draw GH parallel to FC meeting AC produced in H; and draw HI parallel to CB meeting AB in I. Then will G and H be two ſtations and AI the height of the ſpire.

DEMONSTRATION. Join EB, then it is evident from the conſtruction that the points D, A, C, E, F, G, and H, are in the ſame horizontal plane. Now, becauſe HI is parallel to CB, the ∠ AHI is = ACB = 36½°, the angle ſubtended by the ſpire at the middle ſtation H. Again, by ſim. △s, AB : AI : : AC : AH : : AE : AG : : EB : GI; therefore GI is parallel to EB, and therefore ∠ AEB = AGI; but the right-angled triangles EAB, ABD, having BA common, and AD = AE are equal in every *reſpect; hence the* ∠ AGI = ADB = 35¾°, the angle ſubtended by the ſpire at the ſtation G; which, becauſe GH = FC = 35 yards, is = the given diſtance from the ſtation H.

Calculation. The ſides GA, AH of the △ GAH being in the ratio of the cotangents of the given angles, we have cotan. 35¾° : rad. = 1 :: cotan. 36½° : ſine ∠ HGA = 76° 37′ 10″. Then, as coſ. ∠ HGA : GH = 35 yds. :: ſin. ∠ HGA : AH = 147·8 yards. And as rad. : AH :: tan. ∠ AHI : AI = 327·815, to which add 5 feet the height of the eye, gives 332·815 feet the height of the ſpire.

Second Solution, *by Mr.* Gr. Williams, *Caermarthen*.

Fig. 53. pl. 3. Construction. Draw the line AE at right-angles to AB, and from any point *c* in AE, draw *cb* ſo that the angle A*cb* may be equal to 37¼°; from the point *b* where *cb* meets AB, draw *be* ſo as to make the angle A*be* equal to the complement of 36½°, and let it meet AE in *e*. Upon A*c* deſcribe the ſemicircle A*dc*, and apply therein A*d* = A*e*, and join D*c*, in which produced take *d*N = 35 yards, the diſtance between the middle and the extreme ſtations; draw NC parallel to A*d* meeting AC in C; then from C draw CB parallel to *cb* meeting AB in B; ſo will AB be the height of the ſpire above the eye of the obſerver. And if CD be drawn parallel to N*d* meeting A*d* produced in D, then AD will be the diſtance from the laſt ſtation.

Demonstration. Join *de* and take AE = AD; join alſo DE, EB; in CD produced take DG = DC; then will C and G repreſent the two extreme ſtations, whoſe diſtance, (becauſe DCN*d* is a parallelogram, and GD = DC = *d*N = 35) is = the given one, and being at equal diſtances from the foot of the ſpire, the ∠ ACB (= A*cb*) the obſerved altitude at C is = that at G = 35¾° by conſtruction. Again, by parallels A*d* : AD :: A*e* : AE :: A*c* : AC :: A*b* : AB, or A*e* : AE :: A*b* : AB; therefore BE is parallel to *be*, and the angle A*eb* = AEB = ADB (becauſe AE is = AD by conſtr.) = 36½° the given angle.

AD is eaſily found from the above = 147·8 yards, nearly, and AB = 109·3 yards, which being added to the height of the obſerver's eye, gives 332·9 feet for the height of the ſpire required.

Third Solution, *by Mr.* J. Collins, *Kenſington*.

Let A, F, E *(fig.* 54, pl. 3.) be the three ſtations, and let DO *repreſent the height* of the ſpire, excluſive of (5 feet) the height of the obſerver's eye, and draw the lines FD, AD, AC, ED, and EC. Then becauſe the altitudes at the extreme ſtations are equal, thoſe ſtations muſt be equally diſtant from the Church, or ED = AD. Now the right-angled triangles FDC, ADC have DC common to each ∠, DFC = 36° 30′, and ∠ CDA = 35° 45′, therefore, it is manifeſt, that the ſides FD, AD of the right-angled triangle FDA are in proportion as the cotangents of thoſe angles reſpectively, that is, as 1 to 1·0279 nearly, and ſince AF is given = to 35 yards, the method of conſtructing the triangle FAD and determining the height of DC is ſufficiently obvious.

The calculation is alſo extremely eaſy, for AD being to FD as radius to the ſine of DAF, we find that ∠ = 76° 37′ 10″, and ∠ FDA = 13° 22′ 50″, from whence FD is found = $147\frac{4}{5}$, and then in the triangle FDC we find DC = 327·832 feet, and adding 5 feet, we have 332·832 feet for the height of the ſpire.

FOURTH SOLUTION, *by Mr.* JOHN DAWES, *Birmingham.*

Fig. 54, Pl. 3. Let CD repreſent the ſpire, and E, F, A, the three ſtations; then (Euc. I. 47.) FD^2 (x^2) + FE^2 (a^2) = ED^2; that is, ED = $\surd(x^2 + a^2)$. Again, rad. (1) : FD (x) : : tan. DFC (c) : CD = xc, and rad. (1) : ED $(\surd(x^2 + a^2))$: : tan. DEC (b) : CD = $b\surd(x^2 + a^2) = xc$. This equation gives $x = \frac{ba}{\surd(c^2 - b^2)}$ = 147·224 and CD = 110·606 yards.

The ſame was alſo ingeniouſly anſwered by Meſſrs. Cavill, Johnſon, Merones Minor, Milner, Myers, *and* Pickering.

II. QUESTION 32, *by* TYRO PHILOMATHETICUS, *Hull.*

A gentleman has put out 500*l.* at 5 *per cent. per annum*, compound intereſt; and, by agreement, the intereſt is to be payable half yearly. *Query*, what ſum muſt he take out of the ſtock at the end of the firſt half year, ſo that by continuing half yearly the like ſum for 25 years, he ſhall then have received the whole, principal and intereſt.

FIRST SOLUTION, *by Mr.* THOS. MYERS, *Hammerſmith.*

It is manifeſt, from a little conſideration, that this queſtion is no more than to find what annuity payable half-yearly for 25 years may be purchaſed for 500l. allowing compound intereſt at the rate of 5 per cent. Then, from the Theorem on page 168 of Simpſon's Diſſertations, we have the half-yearly payment =

$$\frac{R-1}{1-\frac{1}{R^n}} \times A = 17\text{l. } 12\text{s. } 6\text{d. } 3\tfrac{4}{5}\text{qrs.}; \quad \text{where } A = 500,$$

$R = 1{\cdot}025$, and $n = 50$, the number of payments.

The ſame reſult will alſo be obtained if we find the amount of 500l. at 5 per cent. per ann. compound intereſt for 25 years, and then find what annuity, payable half-yearly, for the ſame time, and at the ſame rate per cent. will be equal to the former amount.

SECOND SOLUTION, *by Mr.* HOPPER.

In this caſe 500l. may be conſidered as the preſent worth of an annuity to continue 25 years; which annuity may be found by Theo. 2, page 238, Simpſon's Algebra. Thus, by uſing the ſame notation, we have $R = 1{\cdot}025$, $n = 50$, $V = 500$; and

$$\log. A = \log. V + \log. (R-1) - \log. \left(1 - \frac{1}{R^n}\right) = 17\text{l.}$$

12s. 7d. nearly, the required annuity.

THIRD SOLUTION, *by Mr.* GEORGE PICKERING.

Let x be the ſum to be deducted at the end of each half-year, $r =$ the ratio, and $a = 500$; then

$ar - x$ .... is the ſum remaining after the 1ſt half-year,
$ar^2 - xr - x$ .... 2d half-year,
$ar^3 - xr^2 - xr - x$ .... 3d half-year,
&c. till

$ar^{50} - xr^{49} - xr^{48}$ .... $xr - x$ ...... 50th half-year.
that is, $ar^{50} - x \times : 1 + r + r^2 + r^3$ ...... $+ r^{49}$ is equal to the ſum remaining after the 50th half-year; but the ſum of the geometrical ſeries

$1 + r + r^2$ .... $+ r^{48} + r^{49}$ is $\frac{r^{50} - 1}{r - 1}$, and

becauſe the whole ſum was to be received at the end of the 50th half-year, therefore, the above expreſſion muſt be equal to nothing, that is, $ar^{50} - x \times \frac{r^{50} - 1}{r - 1} = 0$, or $x \times \frac{r^{50} - 1}{r - 1} = ar^{50}$; conſequently $x = \frac{ar^{50} \times (r - 1)}{r^{50} - 1} = \frac{ar^{51} - ar^{50}}{r^{50} - 1} =$ 17l. 12s. 7d. nearly.

Other Solutions were given by Meſſrs. Cavill, Collins, Johnſon, Marratt, *and* Merones Minor.

III. QUESTION 33, *by Mr.* I. H. HEARDING, *Adderbury.*

$$\text{Given}\begin{cases} x + \frac{x}{1 \cdot 3} + \frac{x}{2 \cdot 3} + \frac{x}{2 \cdot 5} + \frac{x}{3 \cdot 5} + \frac{x}{3 \cdot 7} + \text{\&c. ad infin.} = 30y \\ \frac{1}{3} + y + 2y^2 + \frac{10}{3}y^3 + 5y^4 + \text{\&c. ad infin.} = \frac{9}{10}x \end{cases}$$

to find x and y.

SOLUTION, *by Mr.* WM. MARRATT, *Sibſey, near Boſton.*

By Simpſon's Algebra, page 217, the ſum of the infinite ſeries $x + \frac{x}{1 \cdot 3} + \frac{x}{2 \cdot 3}$ &c. is = to $2x$.

And, multiplying the ſecond equation by 3 we get,

$1 + 3y + 6y^2 + 10y^3 +$ &c. $= \frac{27x}{10}$; that is, by Simpſon's Algebra, page 219, $\frac{1}{(1 - y)^2} = \frac{27x}{10}$ or $\frac{1}{3(1 - y)^3} = \frac{9x}{10}$.

Hence the equations given in the queſtion become,

$$2x = 30y$$

$$\frac{9x}{10} = \frac{1}{3(1-y)^3}.$$

Now, by writing 1 — z inſtead of y in theſe two equations we have

$$x = 15 \times (1-z)$$

$$x = \frac{10}{27z^3}$$

That is, $15 \times (1-z) = \frac{10}{27z^3}$,

$$\text{or } z^4 - z^3 + \frac{2}{81} = 0.$$

Hence, $z = \frac{1}{3}$, $y = \frac{2}{3}$, and $x = 10$.

Nearly in the ſame manner was the Anſwer given by Mr. Hearding, *the Propoſer.*

IV. QUESTION 34, *by Mr.* W. FRANCIS, *Taplow.*

A circular piece of ground, containing 220 acres, is to be divided between three perſons, A, B, C, whoſe ſhares are to be as 5, 6, 7. *Query* the ſhares, by conſtruction, ſo that the boundaries may be equal?

FIRST SOLUTION, *by Mr.* GEORGE PICKERING.

The diameter of the circle is readily found = 131·417 yards. Now let this be divided into three parts, AB, BC, CD, (fig. 55, pl. 3.) having the ſame proportion as the numbers 5, 6, and 7; and on AB, AC, CD, and BD, deſcribe ſemicircles, then the curvilinear ſpaces *aa*, *bb*, *cc*, are the parts required, as is evident from Dr. Hutton's 8th Tract, where this curious diviſion of circles into parts having equal perimeters and *given ratios* was firſt noticed. The content of each ſhare is eaſily found, for $5 + 6 + 7 = 18$, and as 18 is to 220, ſo is each of the numbers 5, 6, 7, to the reſpective ſhares, viz. $61\frac{1}{9}$, $73\frac{1}{3}$, and $85\frac{5}{9}$.

And thus it was anſwered by Meſſrs. Collins, Hopper, Johnſon, *and* Myers.

SECOND

SECOND SOLUTION, *by Mr.* JOHN CAVILL, *Beighton.*

Let PQR (fig. 56, pl. 3.) be the circular piece of ground, the centre of which is O; divide the periphery into 18 equal parts, then take $PQ = 5$, $RP = 6$, and $QR = 7$ of thoſe parts; take alſo $Qd =$ one of thoſe parts, and with the centre d and radius dO deſcribe the arc $Ob =$ arc dQ; again, take Rr $=$ one half of one of the equal parts, and with the centre r and radius rO deſcribe the arc $Oa =$ arc Rr; join ar and bd cutting RO in n and m reſpectively. Then will $PObd =$ A's ſhare, $raObd =$ C's ſhare, and $POar =$ B's ſhare. For the circle is evidently divided in the proportion required by the radii OP, OQ, OR, and the ſpaces mOb, nOa are reſpectively equal to the ſpaces mQd and nrR; and $PQ + Qd + Ob = dr + aO + bO = Pr + aO + Rr =$ ſeven parts; the radius being common to each boundary.

V. QUESTION 35, *by* MERLIN.

To cut 5 equal and ſimilar parts from 5 equal ſquares, ſo that the 10 pieces when joined together ſhall form a ſquare.

SOLUTION, *by* MERLIN, *Jun.*

If 1 be the ſide of each of the 5 equal ſquares, then, into whatever figures they may be cut, the ſide of the greater ſquare muſt be $\sqrt{5}$.

Let ABCD be one of the 5 ſquares: biſect AD in G; then if the ſquare is cut through the line GC it will be divided into a right-angled triangle GDC, and a trapezoid AGCB; and if 4 other like ſquares are cut in the ſame manner, and the parts placed round the ſquare ABCD, (as in fig. 57, pl. 3.) the whole together will form the ſquare EROP. This is too evident from the diagram itſelf to need a formal demonſtration.

Suppoſe $AD = 1$, then $\sqrt{1\frac{1}{4}} = GC = RS = SO$, and $2\sqrt{1\frac{1}{4}} = \sqrt{5} = RO$.

Nearly in the ſame manner was the anſwer given by Meſſrs. **Boole, Cavill, Collins, Dawes, Hopper, Myers,** *and* **Pickering.**

VI. QUESTION 36, *by Mr.* I. T. M'DONELD.

Given the ſum of the baſe and one of the ſides, and the exceſs of the baſe above the other ſide, to conſtruct the plane triangle when its area is either a given ſpace or a maximum.

SOLUTION, *by Mr.* JOHN CAVILL, *Beighton.*

Suppoſe ACB (fig. 58, pl. 3.) to be the triangle required; put AB + BC = s, AB − BC = d, BC = x, AC − AB = y, and the given area = a^2; then, by a well known theorem, $a^2 = \frac{1}{4}\sqrt{\left\{(s^2 - x^2)\ (x^2 - y^2)\right\}}$ or $16a^4 = (s^2 - x^2)\ (x^2 - y^2)$. Now AC = s − AB, and AB = $d + x$, therefore, AC = $s - d - x$, and AC − AB = $s - d - 2x = y$; this value of y being ſubſtituted in the above equation, and then properly reduced, we obtain

$$\left.\begin{array}{l} 3x^4 - 12\,(s - 2d)\,x^3 + 2\,(4d^2 - 4sd - 2s^2)\,x^2 \\ \qquad + s^2\,(4s - 2d)\,x + (s - 2d)^2\,s^2 \end{array}\right\} = \frac{a^2}{16}.$$

From which equation x may be readily determined when the given quantities are expreſſed in numbers.

Again, when the area is a maximum, the fluxion of the above equation muſt be made = 0, and then we have

$$x^3 - (s - 2d)\ x^2 + \frac{1}{6}(4d^2 - 4sd - 2s^2)\,x = \frac{1}{12}\ s^2(4s - 2d).$$

Hence it appears that the problem is not capable of a geometrical conſtruction even in the extreme caſe.

VII. QUESTION 37, *by Mr.* JAMES WHALLEY, *Bolton.*

Having a circle and a triangle ABC, both given in magnitude and poſition: it is required to draw a tangent to the circle, cutting BC in P and AB in Q, so that drawing PR parallel to AB, the ſum, difference, or ratio of the two triangles PQB and *PRC may be given.*

SOLUTION,

SOLUTION, *by* MERONES MINOR.

Suppoſe the line PQK drawn as required, touching the given circle, whoſe centre is O at K; draw DOG parallel to BA, meeting CB produced in G, and PK in D, and on GO demit the perpendicular BL; then by ſimilar triangles, BP : PQ :: GB : BI, and OD : OK :: BI : BL; therefore, by compounding, BP·OD : PQ·OK :: GB : BL. Take *m* a fourth proportional to BL, GB and OK, and then BP·OD $=$ *m*·PQ.

Again, by ſimilar triangles, BP : BQ :: GB : GI, or BQ·GB $=$ BP·GI; therefore, BP·IO $=$ BP·(GO $-$ GI) $=$ BP·GO $-$ BQ·GB, or BP (IO $+$ OD) $=$ BP·BQ $=$ BP·GO $-$ BQ·GB $+$ *m*·PQ; that is, PG·QB $-$ BP·GO $=$ *m*·PQ, or (PG·QB $-$ BP·GO)^2 $=$ m^2·PQ^2. Now, putting $v =$ the verſed ſine of the angle QBP, $a =$ BC, $b =$ BG, $c =$ GO, $x =$ BP, and $y =$ BQ, we have by trigonometry, PQ$^2 =$ $(x \backsim y)^2 + 2vxy$, and the above becomes

$$((b + x)y - cx)^2 = m^2((x \backsim y)^2 + 2vxy) \ldots\ldots \text{E}.$$

Again, becauſe the angle PBQ is given, the triangle QBP has a given ratio to the rectangle BQ·BP; and the triangle RPC being given in ſpecies it will have a given ratio to PC; therefore, when the triangles QBP, RPC, have a given ratio, the rectangle QB·BP will have to PC^2 a given ratio, which ſuppoſe to be that of p to q; then PC $= a - x$, and $xy : (a - x)^2 :: p : q$; now from this equation, and the preceding one, we may obtain the values of x and y by an equation of the 6th power.

Again, when the ſum or difference of the triangles QBP, RPC is given, the ſum or difference of the rectangle QB, BP and a ſpace to which PC^2 has a given ratio is given. Suppoſe this ſpace to be $= n\cdot$PC^2, and that the given ſum or difference is $= s^2$; then $xy \pm n(a - x)^2 = s^2$, and from this equation and the equation E, the values of x and y may be found by an equation of the ſame dimenſions as in the former caſe.

VIII. QUESTION 38, *by Mr.* WILLIAM SMITH, *Liverpool.*

Let AB be the axis of a conic ſection, from B draw BD to meet the curve in D, demit DC perpendicular to AB, and produce it from C till CP is in a given ratio to BD; the locus of the point P will be a conic ſection. Required the demonſtration.

FIRST SOLUTION, *by Mr.* SMITH, *the Propoſer.*

Fig. 60, Pl. 3. Produce PC to H ſo that $BC \cdot CH = CD^2$, and draw AHE to meet BE (drawn to make half a right angle with AB) in E, then becauſe $AC \cdot CB$ has to CD^2 $(BC \cdot CH)$ the ſame ratio as the ſquare of AB to the ſquare of the other axe, the ratio of AC to CH will be given, and, therefore, AE will be given by poſition, and BE is given by poſition, therefore, if EF be drawn perpendicular to AB, BF will be of a given length. Now when AB is the greater axe, AC will be greater than CH, and the point E muſt be below AB. In this caſe $BD^2 = (BC^2 + CD^2 = CQ^2 + CQ \cdot CH) = CQ \cdot QH$, therefore, $CP^2 : CQ \cdot QH$ in a given ratio, but CQ is to CB, and QH to FC, always in a given ratio, therefore, $CP^2 : BC \cdot CF$ is a given ratio, and, therefore, the locus of the point P is a hyperbola.

Again, when AB is the leſs axe, then AC is leſs than C*h*, and the point *e* will be above AB, and by reaſoning as above, we have $CP^2 : fC \cdot CB$ in a given ratio, and, therefore, in this caſe the locus of P is an ellipſe, when the given conic ſection is an ellipſe, but a hyperbola as before when the given ſection is a hyperbola.

Again, when the ſection is a parabola, HE (fig. 61, pl. 3) is parallel to AB, and CH is equal to the parameter of the given parabola, and in this caſe $BC \cdot CF$ has to CP^2 a given ratio, and the locus of P is a hyperbola.

SECOND SOLUTION, *by Mr.* THOMAS BAZLEY, *Bolton.*

I. For the ellipſis (fig. 62, pl. 3.) Put the axis $AB = a$, and its conjugate $2OE' = b$, $BC = x$, and the ratio of $BD : CP :: d : a$. Then, per conics, $a^2 : b^2 :: ax - x^2 : CD^2 = \frac{b^2}{a^2}(ax - x^2)$, conſequently, $DB^2 = \frac{b^2 ax}{a^2} - \frac{b^2}{a^2} x^2 + x^2 = \frac{b^2 \; ax}{a^2} + \frac{a^2 - b^2}{a^2} x^2$, and, therefore, $CP^2 = \frac{a^2}{d^2} \times BD^2 = \frac{b^2 ax + (a^2 - b^2) x^2}{d^2} = \frac{a^2 - b^2}{d^2} \left(\frac{b^2 a}{a^2 - b^2} x - x^2\right)$, which (*when a is greater than b*) is an equation to the hyperbola, whoſe axes are $\frac{b^2}{a^2 - b^2} \times a$ and $\frac{b^2}{d\sqrt{(a^2 - b^2)}} \times a$. But if b is the greater, the equation becomes $CP^2 = \frac{b^2ax - (b^2 - a^2)x^2}{d^2} = \frac{b^2 - a^2}{d^2} (\frac{b^2 a}{b^2 - a^2} x - x^2)$; an equation to the ellipſis, whoſe axes are $\frac{b^2}{b^2 - a^2} \times a$ and $\frac{b^2}{d\sqrt{(b^2 - a^2)}}$. Moreover, if $b = a$, the ellipſis becomes a circle, and the equation for the value of CP becomes $CP^2 = \frac{b^2 a x}{d^2}$; an equation to the parabola, whoſe parameter is $\frac{b^2}{d^2} \times a$.

II. For the hyperbola, (fig. 63, pl. 3.); the ſame notation remaining, $DC^2 = \frac{b^2}{a^2} ax + \frac{b^2}{a^2} x^2$, by the property of the curve; conſequently $CP^2 = \frac{a^2}{d^2} \times BD^2 = \frac{b^2}{d^2} ax + \frac{a^2 + b^2}{d^2} \times x^2 = \frac{a^2 + b^2}{d^2} (\frac{b^2 ax}{a^2 + b^2} + x^2)$, which expreſſes an hyperbola, whoſe axes are $\frac{b^2}{a^2 + b^2} \times a$, and $\frac{b^2}{d\sqrt{(a^2 + b^2)}} \times a$.

III. For the parabola, (fig. 64, pl. 3.). Put the parameter $= p$, then $CD^2 = px$, and $BD^2 = px + x^2$; therefore $CP^2 = \frac{a^2}{d^2} (px + x^2)$, the equation to an hyperbola whoſe axes are p, and $\frac{a}{d} \times p$,

THIRD SOLUTION, *by* MERONES MINOR.

Firſt, let the given conic ſection be a parabola whoſe parameter is BE, (fig. 64, pl. 3.) and take BF : BE in the given ratio of CP : BD; then, by the property of the parabola, $BC \cdot BE = DC^2$; and, therefore, $DB^2 = BC \cdot BE + BC^2 = BC \cdot CE$; wherefore, $BE^2 : BF^2 :: BC \cdot CE : CP^2$, and the locus of the point P is a hyperbola, whoſe vertex is B, and axes BE and BF.

When the given ratio is that of equality, the axes are equal, and in that caſe the hyperbola is equilateral.

Again, let the given conic ſection be an ellipſe, as in (fig. 62, pl. 3.) or an hyperbola, as in (fig. 63, pl. 3.) and BI the reſpective parameters; then, $AB : BI :: AC \cdot BC : DC^2$, or $\frac{BI}{AB} \times AC \cdot BC = DC^2$, and, therefore, $DB^2 = \frac{BI}{AB} \cdot (AB \mp BC) \cdot BC + BC^2 = \frac{BI}{BA} \cdot AB \cdot BC + \frac{AI}{AB} \cdot BC^2$; hence, taking BE a fourth proportional to AI, BI, and AB, we have $AB : AI :: (BE + BC) \cdot BC = BC\ EC : DB^2$. Now, take $BE^2 : BH^2 :: AB : AI$, and $BF : BH$ in the given ratio of $CP : DB$; then, $BE^2 : BH^2 :: BC \cdot EC : DB^2$, and $BF^2 : BH^2 :: CP^2 : DB^2$; therefore, $BE^2 : BF^2 :: BC \cdot CE : CP^2$, and conſequently the locus of P is a hyperbola, whoſe axes are BE and BF, and vertex B. This is general for the hyperbola, whether AB be the greater or leſs axe; but in the ellipſe when AB is the leſs axe it will appear in a manner ſimilar to the above, that the locus will be an ellipſe.

Anſwers were alſo received from Meſſrs. Hopper *and* Johnſon.

IX. QUESTION 39, *by Mr.* THOMAS BAZLEY, *Bolton.*

From the given point P in the periphery of a given ſemicircle APR, it is required to draw the right line PB meeting the diameter AR in B, ſo that BC being drawn perpendicular to AR meeting the periphery in C; $BC^2 - BP^2$ ſhall be of a given magnitude.

FIRST SOLUTION, *by Mr.* J. JOHNSON, *Birmingham.*

CONSTRUCTION. On AR (fig. 65, pl. 3.) demit the perpendicular PD, divide OD at B, (O being the centre) ſo that $2OB \cdot BD$ may be equal to the given difference of the ſquares of BC and BP; draw BP and it is the line required.

Join OP, BC; then, $BC^2 = OC^2 - OB^2$, and $BP^2 = OC^2 (OP^2) - OB^2 - 2OB \cdot BI$ (Eu. 12. II.); therefore, $CB^2 - PB^2$ is $= 2OB \cdot BD$, the given difference. The difference is evidently a maximum when OI is biſected at B, and a minimum *when B coincides with O.*

SECOND

SECOND SOLUTION, *by Mr.* BAZLEY, *the Propoſer.*

CONSTRUCTION. Fig. 65, Pl. 3. Let M^2 = the given magnitude. Then, from the centre O draw OS making the ∠ AOS = $\frac{1}{2}$ a right ∠, and on AR drop the perpendicular PD; from D, with a radius equal to the ſide of the ſquare ($OC^2 - PD^2 - M^2$), deſcribe a circle cutting OS in G and *g*, draw GB, *gb*, perpendicular to AO, and join PB, P*b*, and the thing is done, either of the points B, *b*, anſwering the conditions of the queſtion.

DEMONSTRATION. Join OC and OP. Then, becauſe ∠ BOG = ∠ BGO, BO will be = BG, and $DB^2 + BO^2 = DG^2 = OC^2 - PD^2 - M^2$ per conſtr. whence $M^2 = OC^2 - OB^2 - (PD^2 + DB^2) = BC^2 - PB^2$.

Q. E. D.

COR. If BC be equal to BP, then DG will be equal to DO, and the points B, *b*, will fall in O and D. Alſo, when M^2 is a maximum, GD^2 is evidently a minimum, and in that caſe DG is perpendicular to OS, and the points B, *b*, coincide in the middle of DO.

Ingenious Solutions to this queſtion were alſo received from Meſſrs. Cavill, Hopper, *and* Merones Minor.

X. QUESTION 40, *by* GEOMETRICUS.

Let two circles have the ſame centre, and let A, B, be two given points in the circumference of one of them. It is required to inflect AC, BC, to the circumference of the other circle, ſo that the rectangle AC × CB may be a given magnitude.

FIRST SOLUTION, *by Mr.* THOMAS BAZLEY.

GEOMETRICAL ANALYSIS. Suppoſe the lines AC, CB, (fig. 66, pl. 3.) *drawn as required*, produce AC to meet the exterior circle in D, and join BD. Through the centre O and the point C draw the diameter ECF. Then, ſince the circles are given and concentric, the rectangle EC·CF is given and conſtant. But EC·CF = AC.CD; therefore, AC·CD is given, and by hypotheſis AC·CB is given; conſequently the ratio of AC·CD : AC·CB is given, that is, the ratio of CD : CB is given. But ſince the points A, B are given, the angle ADB is given, and, therefore, the triangle CBD is given in kind; hence the angle BCD as well as its ſupplement ACB is given. Therefore, if we deſcribe on AB the ſegment of a circle to contain the given angle ACB, it will meet the interior circle in the point or points required.

SECOND SOLUTION, *by Mr.* JOHNSON, *Birmingham.*

Through the centre O (fig. 67, pl. 3.) draw the diameter CEOH, meeting the ſmaller circle in E and H; alſo, let AC, BC, meet the ſame circle in D and F, and join AF; then becauſe the given circles are concentric, CE and CH will be given in magnitude, and CB·CF = the given rectangle CE·CH; but by the queſtion the rectangle CA·CB is given, therefore, the ratio of AC to CF is given, and the angle AFC (the ſupplement of the given angle AFB) being given, the triangle ACF will be given in ſpecies, and conſequently the angle ACB will be given; hence if a ſegment of a circle capable of containing this angle be deſcribed on AB, its interſection with the greater of the given circles will determine the point C.

The rectangle AC·CB will be a maximum, when AC = CB, if ABE be the greater ſegment, and a minimum when A*c* = *c*B, but *vice verſa* when ABE is the leſs ſegment. The ſolution will be exactly the ſame when A, B, are in the circumference of the greater circle.

XI. QUESTION 41, *by* HYPATIA.

Three points being given by poſition, it is required to conſtruct a triangle, the ſides of which may be given in magnitude, and paſs through the given points.

SOLUTION, *by Mr.* THOMAS BAZLEY, *Bolton.*

Suppoſe it done, ABC (fig. 68, pl. 3.) the required $\triangle$, D, E, F the given points. Let the points D, E, F be joined: then ſince the ſides of the $\triangle$ ABC are given, the $\angle$s are alſo given; and becauſe DE, DF, and FE are given, the circles deſcribed on thoſe lines to contain the given $\angle$s, C, A, and B, are given in magnitude and poſition. Let DCE, DAF be two of theſe circles, and the queſtion is then reduced to this:

To draw through D, the point of interſection, the right line ADC of a given length, terminated by their peripheries. For then the points A, C, being determined, we have only to draw AFB, CEB, through the points F and E, to meet in B, and the thing is done. Hence the following conſtruction:

Let S, O, the centres of the circles DCE, DAF, be joined, and on OS deſcribe a ſemicircle, in which apply SG $= \frac{1}{2}$AC, and through D draw CDA parallel to SG to cut the circles in C and A; draw CEB, AFB, meeting in B, and ABC is the $\triangle$ required.

DEMONSTRATION. Through O draw OGK to meet AC in K, and SI parallel to OK meeting AC in I; then becauſe $\angle$ G is right, and GS, KI, as alſo GK, SI, parallels, therefore, GI is a rectangle, and SI, GK, are perpendicular to AC; conſequently KD $= \frac{1}{2}$AD, and DI $= \frac{1}{2}$DC, therefore, KD + DI $=$ (KI $=$ GS) $\frac{1}{2}$DA $+ \frac{1}{2}$DC $= \frac{1}{2}$AC: that is, GS $= \frac{1}{2}$AC, and the reſt is plain from the analyſis.

And thus the Queſtion was anſwered by Meſſrs. Collins, Johnſon, Lowry, Merones Minor, Myers, Pickering, *and* Smith.

XII. QUESTION 42, *by* GEOMETRICUS.

A triangle being given by poſition, and two points being alſo given, it is required to draw lines through theſe points to interſect each other in one of the ſides of the triangle, and meet the other ſides ſo as to cut off ſegments adjacent to given points in them, ſuch that the rectangle contained by theſe ſegments may be equal to a given ſpace.

FIRST

FIRST SOLUTION, *by Mr.* LOWRY, *R. M. College.*

ANALYSIS. Let ABC (fig. 69, pl. 3.) be the triangle given by poſition, E and F the given points, through which lines are to be drawn to meet in one of the ſides as AB, and H, L, the given points in the other ſides AC, BC. Now, ſuppoſe the problem reſolved, and that IFG, KEG, are drawn to interſect in AB, and meet AC, BC, in I and K, ſo that the rectangle HI·LK may be equal to a given ſpace. Draw FD parallel to AC, meeting AB in D, and join DE; then DE will evidently be given by poſition. Through the points F, E, draw a line meeting AC in N, and BA produced in O, and draw IQ parallel to OE, meeting GE in Q. Let AQ joined meet GE in M, draw MP parallel to EG, and join NP.

Then, by reaſon of the parallels, AP : PG :: AM : MQ :: AN : NI; therefore, PN is parallel to GFI, and, conſequently, OP : PG :: ON : NF :: OM : ME :: OA : AD; therefore, AM is parallel to DE; but, DE is given by poſition, and A is a given point, therefore, AMQ is given by poſition, and M will be a given point, and NM, ME, given lines. Draw EV making the angle MEV = to the given angle MAN, then EV will evidently be a given line; on OE produced take ET = EV, and draw TR, parallel to ED, and equal to HN; then, becauſe, ET is a given line, and E a given point, T will alſo be a given point, and, therefore, TR, which is parallel to DE will be given by poſition, and alſo in magnitude, being equal to the given line HN, therefore, R is a given point. Now, join RE, and let RT meet GE in S; then, by ſimilar triangles, AM : AN :: ME : EV = ET :: MQ : TS, and AM : AN :: MQ : NI, therefore, TS = NI, and NH is = to TR, therefore, RS is = HI, and, conſequently, the rectangle RS·LK is = to the given rectangle HI·LK; the problem is, therefore, reduced to this; viz. to draw a ſtraight line through E to interſect BC and TR in K and S, ſo that the rectangle of the ſegments LK, RS, adjacent to the given points L, R, may be equal to a given ſpace, which is the ſame as the *Sectio Spatii* of the ancient geometricians. See Dr. Halley's Tranſlation, and alſo the Solution to Q. 220, Old Series of the Repoſitory.

It is obvious that the above analyſis is applicable, when the points K, L, are in the ſide AC inſtead of BC, viz. when the problem requires IFG, *k*EG, to be inflected to AB, meeting AC in I and *k*, ſo that the rectangle IH·*kl* may be equal to a given *ſpace,* H and *l* being given points in the line AC; which is a *caſe that frequently* occurs in the ſolution of problems.

SECOND

SECOND SOLUTION, *by Mr.* W. SMITH, *Liverpool.*

ANALYSIS. Let Aqs (fig. 70, pl. 3.) repreſent the given triangle, d, w the given points in the ſides, and P, Q the other given points; and ſuppoſe QRt, PRT to be drawn as required to interſect Aq in R, and meet ds in t, and As in T, and make the rectangle $wT \times dt =$ to a given ſpace. Join Qd, and draw QI parallel to ds meeting As in I, and IS parallel to Qd, meeting Qt in S, then I will be a given point, and IS will be given by poſition. By the ſimilar triangles Qdt, QIS, we have $dt \times IS =$ the given rectangle $Qd \times QI$, but by the problem the rectangle $dt \times wT$ is given; therefore, the ratio of IS to wT is given. We have, therefore, to inflect two lines QRS, PRT, from the given points Q, P, to interſect Aq in R, and meet IS in S, and AS in T reſpectively, ſo that IS may have to wT a given ratio, which reduces the problem to the ſame as queſtion 17.

Mr. Johnſon *alſo anſwered it.*

XIII. QUESTION 43, *by* SIDROPHEL.

If a triangle given in ſpecies have two of its angles upon ſtraight lines given by poſition and the ſide adjacent to theſe angles paſſing through a given point; required the *Locus* of the angle oppoſite to that ſide.

FIRST SOLUTION, *by Mr.* W. SMITH, *Liverpool.*

ANALYSIS. Let LX, LY (fig. 71, pl. 3.) be the lines given in poſition, O the given point, and ABC the triangle given in ſpecies. Draw BD making the angle $BDC = \angle BLA$, then the ratio of AD to DC is given; alſo the locus of the point D is a right line given in poſition, as LZ, for the points L, A, D, B are in a circle, and, therefore, the angle BLD ($= \angle BAD$) is given. Draw AF, making the angle $LAF = \angle BAD$, and from C draw CF parallel to LD meeting AF in F, then the angle AFC will be given, and if OG be drawn to meet AL in G making the angle OGA = ∠ AFC, then G will be a given point, and the triangles OAG, CAF will be ſimilar; and, therefore, AG : OG :: AF : FC, or AG·FC = OG·AF. From G draw G*k*K, meeting LF in *k*, and CF in K, then if AF meet LD in *f*, the ratio of AD : DC being given, that of AF : *f*F will be given, and the locus of F will be a right line given in poſition, as LF; therefore, AG has to *k*F a given ratio; viz. that of GL to L*k*, and AF has to LF a given ratio alſo, viz. that of G*k* to *k*L, and OG being a given line, the rectangle *k*F·FC will be equal to the rectangle under LF and a given line, equal, ſuppoſe to LF·LR, and, therefore, the parallelogram *k*FCH = the parallelogram LFR*r*, or the parallelogram I*r*CH = the parallelogram LKIR, that is, the rectangle I*r*·*r*C (= L*k*·*k*I) is given; and, therefore, the locus of C will be a hyperbola, whoſe aſymptotes are I*r* and IH.

SECOND SOLUTION, *by Mr.* LOWRY, *R. M. College.*

Let ABC (fig. 72, pl. 3.) be the triangle given in ſpecies, having the angles A, B, on the ſtraight lines EA, EB given by poſition, and the ſide AB paſſing through the given point P. Draw EH to make the angle BEH equal to the given angle BAC, then EH will be given by poſition. Conceive CH and BH to be drawn ſo as to make the angle CHB equal to the given angle BEA, and join BK, K being the point where CH meets AE; then, becauſe, the angles KHB, KEB are equal, the points E, H, K, B are in a circle; therefore, the angles BKC, BEH, BAC are equal, and, conſequently, the points B, K, A, C are alſo in a circle; and, therefore, the angles EKB, (EHB), ACB are equal; the triangle EBH is, therefore, ſimilar to ACB, and if PF be drawn parallel to AE, and FG parallel to BH, the angle EFG will be equal to the angle ABC, and FG, EG will be given in magnitude; and, moreover, if GL be drawn parallel to CH it will be given by poſition, becauſe the angle FGL is equal to the given angle BEA.

Again, becauſe PF is parallel to AE, the angle PFB is equal to AEB (CHB), and the angle PFB is equal to CBA, therefore, adding the common angle APH, the triangles BFP, BHC, appear to be equiangular, and, conſequently, BF : BH :: PF : CH; but the ratio of BF : BH is compounded of the ratios of HG : HE, and FG : EF; wherefore, PF : CH :: HG·FG : HE·EF. Take IG : PF :: EF : FG, then IG : CH :: HG : *HE*; and, by diviſion, CD : IG :: HG (ID) : EG (ID being *drawn parallel to EH*, meeting CH in D); therefore, CD·DI is $=$ GI·GE $=$ to a given ſpace, and the locus of the point C is a hyperbola, whoſe aſymptotes are IL and ID.

Mr. Johnſon *likewiſe anſwered this Queſtion.*

XIV. QUESTION 44, *by Mr.* JOHN WRIGHT, *Norley.*

To find three ſquare numbers in harmonical proportion ſuch, that, if from each of them its reſpective root be taken, the remainders thence ariſing ſhall be rational ſquare numbers.

SOLUTION, *by the Propoſer.*

Let x^2, y^2, and z^2 be the three required numbers. Then by the property of harmonical proportion,

$x^2 - y^2 : y^2 - z^2 :: x^2 : z^2$; hence, $y^2 = \frac{2x^2 z^2}{x^4 + z^2}$,

or $y = \frac{xz\sqrt{2}}{\sqrt{(x^2 + z^2)}} = \frac{(r^2 - s^2 + 2rs)\ (s^2 - r^2 + 2rs)\ l}{r^2 + s^2}$,

as will appear by ſubſtituting

$\frac{(r^2 - s^2 + 2rs)\ l}{r^2 + s^2}$ for x, and $\frac{(s^2 - r^2 + 2rs)\ l}{r^2 + s^2}$ for y.

Aſſume now

$l = \frac{(r^2 + s^2)\ n^2}{(2n - m)\ n}$, and putting $a = r^2 - s^2 + 2rs$;

$b = s^2 - r^2 + 2rs$, and $c = r^2 + s^2$; the expreſſions for x, y, and z above, will be transformed into

$\frac{n^2}{(2n - m)\,m}$, $\frac{bn^2}{(2n - m)\ cm}$, and $\frac{b^2\,n^2}{(2n - m)\ am}$ reſpectively.

Now, per queſt. $x^2 - x$, $y^2 - y$, and $z^2 - z$ are all to be ſquare numbers:

but $x^2 - x = \frac{n^2\ (n + m)^2}{m^2\,(2n - m)^2}$ is already a ſquare number,

and $y^2 - y = \frac{n^2}{(2n - m)^2\ c^2\,m^2} \times \left\{b^2\,n^2 - (2n - m)\,bcm\right\}$

will be a ſquare, when $b^2\,n^2 - (2n - m)\ bcm$ is ſo: to effect which, aſſume $b^2 n^2 - (2n - m) \cdot bcm = (bn - pm)^2$; which will give $m = 2bn \times \frac{c - p}{bc - p^2}$. This value of m being written inſtead of it in the expreſſion for z laſt found, we ſhall get

$$z = \frac{(bc - p^2)^2}{4ap\,(c - p)\,(b - p)},$$ and hence

$$z^2 - z = \frac{(bc - p^2)^2}{16a^2 p^2 (c - p)^2 (b - p)^2} \times$$

$$\left\{(bc - p^2)^2 - 4ap\,(b - p)\,(c - p)\right\}$$

which muſt be a ſquare number, conſequently $(bc - p^{2 \cdot 2} - 4ap\,(b - p)\,(c - p) = b^2 c^2 - 4apbc - (2bc - 4ac - 4ab)\,p^2 - 4ap^3 + p^4 =$ a ſquare $= (bc + 2ap - p^2)^2$ ſuppoſe; whence is obtained

$$p = \frac{2bc}{c + b - a} = 2 \times \frac{s^4 - r^4 + 2rs\,(r^2 + s^2)}{3s^2 - r^2}$$

where r and s may be expounded by any two numbers taken at pleaſure.

Ex. Gr. Take $r = 2$, and $s = 3$; then

$$m = \frac{111826n}{78455},$$ alſo $$x = \frac{6155187025}{5041563384},$$

$$y = \frac{6155187025}{3855313176},$$ and $$z = \frac{6155187025}{2075937864}.$$

XV. QUESTION 45, *by* X. Y.

It is required to inveſtigate a rule for finding the area of any polygon whatever by means of its ſides and angles.

SOLUTION, *by Mr.* LOWRY.

The inveſtigation of a general rule for finding the area of any *rectilineal* polygon, by means of its ſides and angles, will be much *facilitated by the following* Lemmas:

1ſt. Let a, b, c be any three arcs, then,

$$\text{ſin}\,(b+c) = \frac{\text{ſin}\,a\ \text{ſin}\,c + \text{ſin}\,b\ \text{ſin}\,(a+b+c)}{\text{ſin}\,(a+b)}$$

It is well known that the $\text{ſin}\,(b+c) = \text{ſin}\,b\ \text{coſ}\,c + \text{coſ}\,b\ \text{ſin}\,c$; and, therefore, $\text{ſin}\,(a+b+c) = \text{ſin}\,a\ \text{coſ}\,(b+c) + \text{coſ}\,a\ \text{ſin}\,(b+c) =$ (if $\text{coſ}\,b\ \text{coſ}\,c - \text{ſin}\,b\ \text{ſin}\,c$ be put for $\text{coſ}\,(b+c)$) $\text{ſin}\,a\ \text{coſ}\,b\ \text{coſ}\,c - \text{ſin}\,a\ \text{ſin}\,b\ \text{ſin}\,c + \text{coſ}\,a\ \text{ſin}\,(b+c)$; and, multiplying by $\text{ſin}\,b$ and adding $\text{ſin}\,a\ \text{ſin}\,c$, we have $\text{ſin}\,a\ \text{ſin}\,c + \text{ſin}\,b\ \text{ſin}\,(a+b+c) = \text{ſin}\,a\ \text{coſ}\,b\ \text{coſ}\,c\ \text{ſin}\,b + \text{ſin}\,a\ \text{ſin}\,c\ \text{coſ}^2 b + \text{coſ}\,a\ \text{ſin}\,b\ \text{ſin}\,(b+c) = \text{ſin}\,a\ \text{coſ}\,b\ (\text{ſin}\,b\ \text{coſ}\,c + \text{coſ}\,b\ \text{ſin}\,c = \text{ſin}\,(b+c) + \text{coſ}\,a\ \text{ſin}\,b + \text{ſin}\,(b+c) = \text{ſin}\,a\ \text{coſ}\,b + \text{coſ}\,a\ \text{ſin}\,b)\ (\text{ſin}\,(b+c)) = \text{ſin}\,(a+b)\ \text{ſin}\,(b+c)$; therefore, it is manifeſt that

$$\text{ſin}\,(b+c) = \frac{\text{ſin}\,a\ \text{ſin}\,c + \text{ſin}\,b\ \text{ſin}\,(a+b+c)}{\text{ſin}\,(a+b)}.$$

It might be ſhewn in a ſimilar manner that

$$\text{ſin}\,(b-c) = \frac{\text{ſin}\,a\ \text{ſin}\,c - \text{ſin}\,b\ \text{ſin}\,(a-b+c)}{\text{ſin}\,(a-b)}.$$

2d. Let a, b, c, d be four arcs, then

$$\text{ſin}\,(b+c+d) = \frac{\text{ſin}\,a\ \text{ſin}\,(c+d) + \text{ſin}\,b\ \text{ſin}\,(a+b+c+d)}{\text{ſin}\,(a+b)}.$$

This is immediately deduceable from the 1ſt Lemma, by writing $(c+d)$ inſtead of c, and by that means may be extended to any number of arcs.

3d. If there are five arcs a, b, c, d, e, then

$$\text{ſin}(b+c+d+e) = \frac{\text{ſin}\,a\ \text{ſin}\,(c+d+e) + \text{ſin}\,b\,\text{ſin}\,(a+b+c+d+e)}{\text{ſin}\,(a+b)}.$$

And in general for any number of arcs a, b, c, &c. ...... n,

$$\text{ſin}\,(b+c+\ldots.n) = \frac{\text{ſin}\,a\ \text{ſin}(c+d+\ldots.n) + \text{ſin}\,b\,\text{ſin}(a+b+c\,..n)}{\text{ſin}\,(a+b)}$$

To proceed with perſpicuity we ſhall firſt find the area of a trapezium or polygon of four ſides. Let two of the ſides, as AB, DC (fig 73, pl. 3.) be produced till they meet at b; then it is evident that the trapezium ABCD is equal to the difference of the triangles AbD, BbC. Now twice the area of the triangle AbD is $= \text{A}b\cdot b\text{D} \cdot \text{ſin}\,b = (\text{AB} + \text{B}b)\,(\text{DC} + \text{C}b)\ \text{ſin}\,b$, and twice the area of the triangle BbC $= \text{B}b\cdot\text{C}b\cdot\ \text{ſin}\,b$; and, therefore, their difference or twice the area of the trapezium is $= (\text{AB}\cdot\text{DC} + \text{AB}\cdot\text{C}b + \text{DC}\cdot\text{B}b)\ \text{ſin}\,b$; but $\text{C}b\cdot\ \text{ſin}\,b$ is $= \text{BC}$ ſin B, and ſin BC · ſin b = BC · ſin C, and ſin b = the ſine of the ſum of the *exterior* angles $B + C$, therefore, twice the area of the trapezium is =

$$\begin{aligned} & AB\cdot BC\ \text{ſin}\,B \\ + & AB\cdot DC\ \text{ſin}\,(B+C) \\ + & BC\cdot DC\ \text{ſin}\,C. \end{aligned}$$

Cor. Twice the area of the triangle ABC is =

$$\begin{aligned} & AB\cdot DC\cdot \text{ſin}\,(B+C) \\ + & BC\cdot DC\cdot \text{ſin}\,C. \end{aligned}$$

Again, the area of the pentagon ABCDE (fig. 74, pl. 3.) is equal to the ſum of the areas of the trapezium ABCD, and the triangle DAE. Let the ſides AB, DC be produced till they meet at *b*, then from the above we have twice the area of the trapezium ABCD =

$$\begin{aligned} & AB\cdot BC\cdot \text{ſin}\,B \\ + & AB\cdot DC\cdot \text{ſin}\,(B+C) \\ + & BC\cdot DC\cdot \text{ſin}\,C. \end{aligned}$$

And twice the area of the triangle DAE is =

$$\begin{aligned} & Ab\cdot DE\cdot \text{ſin}\,(b+D) \text{ or ſin.}\,(B+C+D) \\ + & Db\cdot DE\cdot \text{ſin}\,D. \text{ That is } = \\ & AB\cdot DE\cdot \text{ſin}\,(B+C+D) \\ + & DC\cdot DE\cdot \text{ſin}\,D \\ + & Bb\cdot DE\cdot \text{ſin}\,(B+C+D) \\ + & Cb\cdot DE\cdot \text{ſin}\,D. \end{aligned}$$

But $Bb = \dfrac{BC\cdot \text{ſin}\,C}{\text{ſin}\,(B+C)}$, and $Cb = \dfrac{BC\cdot \text{ſin}\,B}{\text{ſin}\,(B+C)}$, therefore, the laſt two terms together become

$BC\cdot DE\,(\text{ſin}\,C\cdot \text{ſin}\,(B+C+D) + \text{ſin}\,B\ \text{ſin}\,D) = \text{ſin}\,(B+C)$ (by the 1ſt Lemma) $BC\cdot DE\cdot \text{ſin}\,(C+D)$. Now the various terms being collected, and arranged according to the order of the ſides, we have twice the area of the pentagon =

$$\begin{aligned} & AB\cdot BC\cdot \text{ſin}\,B \\ + & AB\cdot DC\cdot \text{ſin}\,(B+C) \\ + & AB\cdot DE\cdot \text{ſin}\,(B+C+D) \\ + & BC\cdot DC\cdot \text{ſin}\,C \\ + & BC\cdot DE\cdot \text{ſin}\,(C+D) \\ + & DC\cdot DE\cdot \text{ſin}\,D. \end{aligned}$$

The ſame reſult will be obtained with equal facility by finding the difference between the trapeziuum A*b*DE and the triangle B*b*C.

Cor. Twice the area of the triangle DAE is =

$$\begin{aligned} & AB\cdot DE\cdot \text{ſin}\,(B+C+D) \\ + & BC\cdot DE\cdot \text{ſin}\,(C+D) \\ + & DC\cdot DE\cdot \text{ſin}\,D. \end{aligned}$$

In a ſimilar manner we obtain the area of a hexagon by finding the ſum of the areas of the pentagon ABCDE (fig. 75, pl. 3.) and the triangle AFE; for by the laſt rule twice the area of the pentagon is =

$$\begin{aligned}
&\text{AB}\cdot\text{BC}\cdot\text{ſin B}\\
+\;&\text{AB}\cdot\text{CD}\cdot\text{ſin}\,(B+C)\\
+\;&\text{AB}\cdot\text{DE}\cdot\text{ſin}\,(B+C+D)\\
+\;&\text{BC}\cdot\text{CD}\cdot\text{ſin C}\\
+\;&\text{BC}\cdot\text{DE}\cdot\text{ſin}\,(C+D)\\
+\;&\text{CD}\cdot\text{DE}\cdot\text{ſin D}.
\end{aligned}$$

And twice the area of the triangle AEF =

$$\begin{aligned}
&\text{A}b\cdot\text{EF}\cdot\text{ſin}\,(B+C+D+E)\\
+\;&\text{D}b\cdot\text{EF}\cdot\text{ſin}\,(D+E)\\
+\;&\text{DE}\cdot\text{EF}\cdot\text{ſin E}\\
\text{equal }&\text{AB}\cdot\text{EF}\cdot\text{ſin}\,(B+C+D+E)\\
+\;&\text{DC}\cdot\text{EF}\cdot\text{ſin}\,(D+E)\\
+\;&\text{DE}\cdot\text{EF}\cdot\text{ſin E}\\
+\;&\text{B}b\cdot\text{EF}\cdot\text{ſin}\,(B+C+D+E)\\
+\;&\text{C}b\cdot\text{EF}\cdot\text{ſin}\,(D+E).
\end{aligned}$$

and writing for B*b* and C*b* their reſpective values $\dfrac{\text{BC}\cdot\text{ſin C}}{\text{ſin}\,(B+C)}$, and $\dfrac{\text{BC}\cdot\text{ſin B}}{\text{ſin}\,(B+C)}$ the ſum of the laſt two expreſſions becomes

$$\text{BC}\cdot\text{EF}\cdot\text{ſin C}\cdot\text{ſin}\,\frac{(B+C+D+E)+\text{ſin}\,(D+E)\ \text{ſin B}}{\text{ſin}\,(B+C)} =$$

(by the 2d Lemma) to BC·EF · ſin (C + D + E), and collecting and properly arranging the ſeveral terms we have twice the area of the hexagon =

$$\begin{aligned}
&\text{AB}\cdot\text{BC}\cdot\text{ſin B}\\
+\;&\text{AB}\cdot\text{CD}\cdot\text{ſin}\,(B+C)\\
+\;&\text{AB}\cdot\text{DE}\cdot\text{ſin}\,(B+C+D)\\
+\;&\text{AB}\cdot\text{EF}\cdot\text{ſin}\,(B+C+D+E)\\
+\;&\text{BC}\cdot\text{CD}\cdot\text{ſin C}\\
+\;&\text{BC}\cdot\text{DE}\cdot\text{ſin}\,(C+D)\\
+\;&\text{BC}\cdot\text{EF}\cdot\text{ſin}\,(C+D+E)\\
+\;&\text{CD}\cdot\text{DE}\cdot\text{ſin D}\\
+\;&\text{CD}\cdot\text{EF}\cdot\text{ſin}\,(D+E)\\
+\;&\text{DE}\cdot\text{EF}\cdot\text{ſin E}.
\end{aligned}$$

The ſame reſult may alſo be obtained by taking the difference of the areas of the pentagon A*b*DEF, and the triangle B*b*C; and we have alſo this

Cor. Twice the area of the triangle AEF is =

$$\begin{aligned}
&\text{AB}\cdot\text{EF}\cdot\text{ſin}\,(B+C+D+E)\\
+\;&\text{BC}\cdot\text{EF}\cdot\text{ſin}\,(C+D+E)\\
+\;&\text{CD}\ \text{EF}\cdot\text{ſin}\,(D+E)\\
+\;&\text{DE}\cdot\text{EF}\cdot\text{ſin E}.
\end{aligned}$$

In a like manner may the area of a heptagon be found, by finding the ſum of the areas of the hexagon and the adjacent triangle, and from thence the area of an octagon, and ſo on for any other polygon.

The law of continuation is, however, ſufficiently obvious from what has been done above, to enable us to extend the rule to any polygon whatever, without further inveſtigation, and it may be expreſſed as follows:

Twice the area of any polygon is equal to the ſum of the rectangles of its ſides, except one, taken two and two, by the ſines of the ſums of the *exterior* angles contained by theſe ſides.

In the application of the rule to any practical purpoſe, it will be moſt convenient to divide the polygon into two parts by a diagonal, and compute the area of each part ſeperately.

Thus if the trapezium be divided into two triangles by the diagonal AC, twice the area will be equal to

$$\begin{aligned} &AB\cdot BC\cdot \text{ſin}\,B\\ +\,&CD\cdot AD\cdot \text{ſin}\,D.\end{aligned}$$

And if the pentagon be divided into the trapezium ABCD, and the triangle ADE, we have twice the area =

$$\begin{aligned} &AB\cdot BC\cdot \text{ſin}\,B\\ +\,&AB\cdot DC\cdot \text{ſin}\,(B+C)\\ +\,&BC\cdot DC\cdot \text{ſin}\,C\\ +\,&DE\cdot AE\cdot \text{ſin}\,E.\end{aligned}$$

In like manner the hexagon may be divided into two trapeziums, and then twice the area will be =

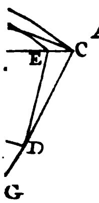
$$\begin{aligned} &AB\cdot BC\cdot \text{ſin}\,B\\ +\,&AB\cdot DC\cdot \text{ſin}\,(B+C)\\ +\,&BC\cdot DC\cdot \text{ſin}\,C\\ +\,&DE\cdot EF\cdot \text{ſin}\,E\\ +\,&DE\cdot AF\cdot \text{ſin}\,(E+F)\\ +\,&EF\cdot AF\cdot \text{ſin}\,F.\end{aligned}$$

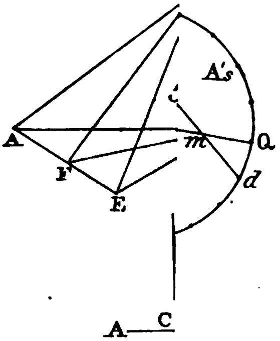
XVI. QUESTION 46, *by* NAUTICUS.

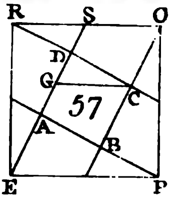
Suppoſe a great circle to paſs through a given point of the equator and make with it a given angle, it is required to find its *projection* on Mercator's Chart.

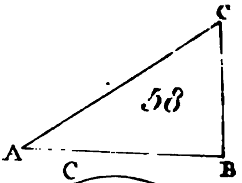
SOLUTION.

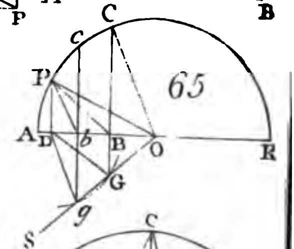
Plate III. Fig 52 to

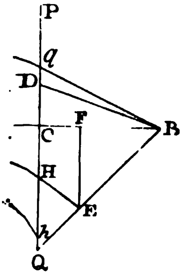


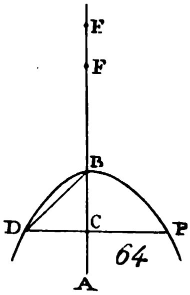


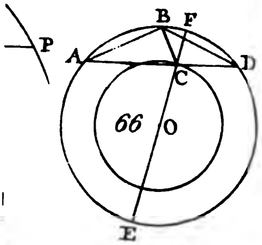


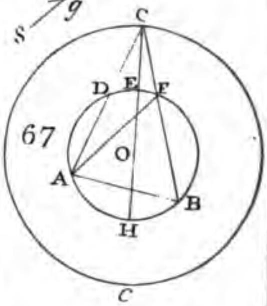


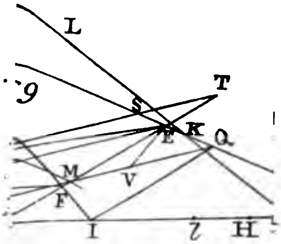


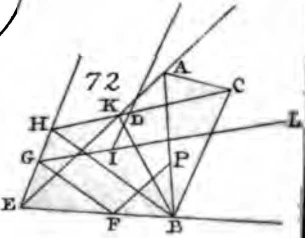


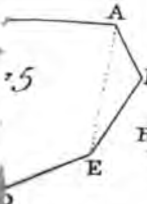


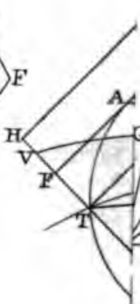


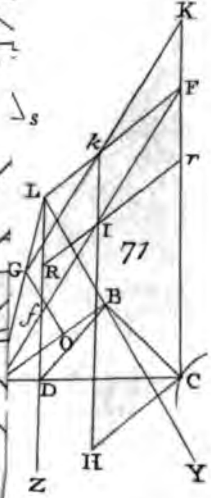


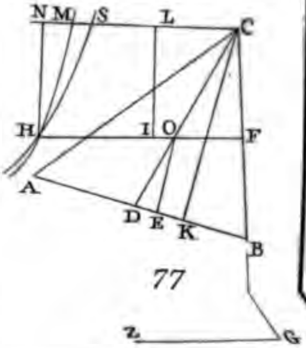












SOLUTION, *by Mr.* WALLACE, *R. M. College.*

Let the curve, whoſe nature is ſought, be referred to the ſtraight line repreſenting the equator, as an axis; and let the origin of the abſciſſes be at the given point in which the great circle meets the equator.

Let α denote the given angle; let φ denote an arch of the equator between the given point and a meridian paſſing through any variable point in the great circle, and ψ the latitude of that point: put x and y for the ſtraight lines in the chart which repreſent the arches φ and ψ, or for the co-ordinates of a point in the curve whoſe latitude is ψ and longitude φ. Then, from the nature of the projection,

$$x = \varphi, \quad \dot{y} = \frac{\dot{\psi}}{\text{Coſ.}\ \psi}.$$

And from ſpherical trigonometry,

$$\tan.\ \psi = \tan.\ \alpha.\ \text{ſin.}\ \varphi;$$

and from theſe equations we are to deduce the relation between x and y.

Taking the fluxion of this laſt equation, and putting $\tan.\ \alpha = a$, we have

$$\frac{\dot{\psi}}{\text{Coſ.}^2\ \psi} = a\ \dot{\varphi}\ \text{coſ.}\ \varphi, \text{ and, therefore, } \dot{y} = a\ \dot{\varphi}\ \text{coſ.}\ \varphi\ \text{coſ.}\ \psi.$$

But from the equation, $\tan.\ \psi = a\ \text{ſin.}\ \varphi$, we find

$$\text{Coſ.}\ \psi = \frac{1}{\sqrt{(1 + a^2\ \text{ſin.}^2\ \varphi)}}.$$

$$\text{Therefore, } \dot{y} = \frac{a\ \dot{\varphi}\ \text{coſ.}\ \varphi}{\sqrt{(1 + a^2\ \text{ſin.}^2\ \varphi)}},$$

and taking the fluents ſo that $y = 0$, when x or $\varphi = 0$,

$$y = \text{hyp. log.} \left\{ a\ \text{ſin.}\ \varphi + \sqrt{(1 + a^2\ \text{ſin.}^2\ \varphi)} \right\}.$$

Hence the nature of the curve may be expreſſed thus:
Let φ be any arch of a circle to radius 1.
Take $x = \varphi$,

$$y = \text{hyp. log.} \left\{ a\ \text{ſin.}\ \varphi + \sqrt{(1 + a^2\ \text{ſin.}^2\ \varphi)} \right\}.$$

Then x and y are co-ordinates to a point in the curve.

XVII. QUESTION 47, *by* GEOMETRICUS.

Let A, B be two given points in AB a ſtraight line given by poſition, and let C, D be two given points without that line; let CV, DV be drawn, meeting AB in F and G, ſo that the rectangle AF × BG is given: It is required to determine the Locus of the point V.

SOLUTION, *by Mr.* LOWRY.

ANALYSIS. Join AC, BD, (fig. 78, pl. 4.) and draw DE parallel to AC meeting CI drawn parallel to AB in I. Let the rectangle AC·CR be taken equal to the given rectangle AF·BG, draw SRQ parallel to AC, and take RQ = EB; draw CQ to meet EI produced in K, and DP parallel to AB meeting AC in P; alſo draw VL parallel to AC meeting PD in H, CI in O, and CK in L; then the triangles ACF, CRS, being ſimilar, AC·CR = AF·RS, and by the hypothesis AC·CR = AF.BG, therefore RS is = BG, and RQ is = BE, therefore, SQ is = EG. Again, by ſimilar triangles, DE : EG = SQ :: VH : HD, and SQ : CR :: VL : CO = PH; therefore, DE : CR :: VH·VL : DH·HP. Now in the trapezium CPDK, VL is drawn parallel to the ſides CP, DK, and PD parallel to another of the ſides AE, and the rectangles of the ſegments VH·VL, DH · HP are in a given ratio, namely, that of DE to CR; therefore, the *locus* of the point V is a conic ſection, (ſee Hamilton's Conics, p. 56, B. i.) and is the celebrated *Locus Solidus* of the Ancient Geometricians. Newton in his Principia has given a ſolution to the general caſe of this Problem, and enumerated ſeveral of the particular ones, and the Problem has been ſtill more diſtinctly conſidered by Dr. Pemberton, in an ingenious paper printed in the Philoſophical Tranſactions for the year 1763. In what follows we ſhall endeavour to determine the ſpecies of the Locus in as brief and general a manner as poſſible, without entering into a particular detail of the various caſes for each ſection; and in order to do this, let DP, KC, be produced till they meet at X, and XM be drawn to biſect CP, DK at Z and N, and meet the ſection in T and M; then it *is evident that* TM will be a diameter of the ſection, and VY *an ordinate to that diameter*. Now let MW be drawn parallel to VY, meeting XD produced in W; then MW will be a tangent to the curve at M, and by the general property of the ſections, the rectangles $VH\cdot VL$, $DH\cdot HP$, are as the ſquares of the ſemiaxes parallel to VL and PD, and (Hamilton, p. 31, B. i.) MW^2 is to $DW\cdot WP$ in the ſame ratio; therefore, $MW^2 : DW\cdot WP\ (VH\cdot VL : DH\cdot HP) :: DE : CR :: DE\cdot CR : CR^2$. Again, by ſimilar triangles, $WX : 2MW :: CI : IK$; therefore, $WX^2 : MW^2 :: 4\cdot CI^2 : IK^2 :: 4\cdot CR^2 : RQ^2 = BE^2$, and *(ex aequo)* $WX^2 : DW\cdot WP :: DE\cdot CR : \frac{BE^2}{4}$.

Now it is evident from the figure of the ſections, that in the ellipſe WX^2 is greater than $DW\cdot WP$, in the parabola equal, and in the hyperbola leſs; therefore the locus will be an ellipſe parabola, or hyperbola, according as $DE\cdot CR$ is greater, equal to, or leſs than the ſquare of half BE.

When the points BE coincide, the lines CI and CK will coincide, and the trapezium will become a parallelogram, and in this caſe the locus will be an ellipſe, except when AC, DB, are perpendicular to AB, and the given rectangle $= AC\cdot DB$, and then it becomes a circle whoſe diameter is CD, and from thence we derive the following properties of the circle.

1st. If from the extremities CD of the diameter of a circle, (fig. 79, pl. 4.) any two lines CV, DV, be inflected to the circumference, meeting the ſide AB of a rectangle deſcribed on CD in F and G: then the rectangle $AF\cdot BG$ will be equal to the ſquare of AC or BD.

2dly. If from CD, the ends of the diameter, (fig. 80, pl. 4.) any two lines CV, DV, be inflected to the circumference, meeting any ſtraight line in F and G, and the perpendiculars AC, DB be drawn to AB, then $AF\cdot BG$ will be $= CA\cdot DB$.

There is alſo another caſe in which the locus is a circle, namely, when the diameter VM is perpendicular to the parallels AC, DK, and the given rectangle = to $AC\cdot DE$. In this caſe (fig. 81, pl. 4.) PD is = CK, and if VY be produced to meet the circle again in *a*, VY will be $= aY$, $VH = aL$, and $VL = aH$; therefore, $PH\cdot HD = VH\cdot Ha = VH\cdot VL$, as it ought to be. From the above we deduce the following curious theorem, viz.

If two equal chords PD, CK, be inſcribed in a circle, and the lines CP, KD joining their extremities, be produced to meet any ſtraight line AB drawn parallel to PD in A and E, and DB be drawn, making the angle EDB equal to the angle made by the chords produced till they meet; then if any two lines CV, DV, be inflected to the circumference meeting AE in F and G, the rectangle $AF\cdot BG$ will be equal to $CA\cdot DE$.

There is alſo a caſe in which the locus becomes a ſtraight line, viz. when the line joining the points CD, (fig. 82, pl. 4.) interſects AB at E, making the rectangle AE·BE equal to the given one. In this caſe draw ICK parallel to AB, and let the line joining BD meet CKI in K; take AH = BE, and draw KHV, which will be the locus ſought. For join AC, and produce VD to meet KC at I; then by reaſon of the parallels AH = BE : BG :: KC : KI :: HF : HG; therefore, by permutation and compoſition, AF : AH = BE :: HB = AE : BG; and therefore AF·BG is = AE·EB.

Meſſrs. Hopper *and* Smith *alſo anſwered this Queſtion.*

XVIII. QUESTION 48, *by Mr.* W. SMITH, *Liverpool.*

Given the perimeter, the area, and the difference of the angles at the baſe of a plane triangle to conſtruct it by the conic ſections.

FIRST SOLUTION, *by the Proposer.*

CONSTRUCTION. Fig, 76, pl. 3. Form a rectangle HNMT ſuch that its longeſt ſide HN may be equal to half the given perimeter, and its area equal to twice that of the given triangle; through the middle points of HT, NM draw CF, and through the point T with the aſymptotes CF, CM deſcribe the equilateral hyperbola TS. Divide TH in V ſo that TV may be to VH in the ratio of the line biſecting the vertical angle to the perpendicular, which ratio is given, becauſe the difference of the angles at the baſe is given; now with the focus T, vertex V, and directrix HN, deſcribe the hyperbola VS interſecting the former one in S, draw SQ parallel to CF meeting TF produced in Q, and join QC meeting TM in O, and on OQ deſcribe a circle cutting FC in A, and CB drawn to make the angle QCB = ACQ in B; then if AB be drawn, ACB will be the triangle required.

DEMONSTRATION. By conſtruction, the points A, O, B, Q are in a circle whoſe diameter is OQ; and the angle ACB is biſected by CQ, therefore, by a well known property of plane triangles, O is the centre of the inſcribed circle, and Q the centre of a circle touching the baſe and the continuation of the two ſides of the triangle ACB, and, therefore, the perimeter is equal to twice CF. Let SO produced meet CF in G, then SG is equal and parallel to FQ, and OG equal to the radius of the inſcribed circle; and becauſe the angles OSQ, OTQ are right ones, the points T, S are in the circumference of the circle deſcribed on OQ, and if TS be drawn, it will be a diameter, and therefore equal to QO.

Now by the property of the hyperbola and the conſtruction OQ or TS : HQ : : TV : VH. Draw CD and QW perpendicular to AB, the latter meeting the circle in X, and join OX; let I be the point where AB meets CQ. Then it is well known that QX is = to QH; and, therefore, OQ : QX :: IC : CD :: TV: VH, that is, IC, the line biſecting the vertical angle, has to CD, the perpendicular, the given ratio; alſo the area of the triangle ACB is equal to the rectangle FC × OG, or FC × CM, that is, equal to the given ſpace. *Q. E. D.*

SECOND SOLUTION, *by Mr.* LOWRY.

ANALYSIS. Let ABC (fig. 77, pl. 3.) be the triangle required, DC the line biſecting the vertical angle, and O the centre of the inſcribed circle; draw the radii OE, OF perpendicular to AB, BC, and on CB produced take CG = to half the given perimeter, and draw CK perpendicular to AB; then it is well known that FG = AB, and AB·CK = 2OF (or 2OE)· CG, therefore OF = OE is given in magnitude; moreover the angle DOE = DCK = half the difference of the angles at the baſe is given, and, therefore, OD is given in magnitude; and the ratio of DC to CK being the ſame as the ratio of DO to OE, it will be given; and, therefore, AB·DC will be = to 2OD·CG = to a given ſpace = to MC·CG (if CM be drawn perpendicular to CG and = to 2OD). Now let FO be produced till FH = DC, and then BF·FH = AB·DC = CG·MC, therefore the locus of the point H is a hyberbola paſſing through M, and having CG and a line GZ drawn perpendicular to CG for its aſymptotes. Again, biſect CM at L, and draw LI parallel to CB, meeting FH at I, and HN parallel to IL, meeting CM produced in N, and take LS = OF, then NH = CF and LN = IH = OC, and, therefore, $LN^2 - LS^2 = HN^2$ or $2LN \cdot LS = HN^2$, and consequently the locus of H is another equilateral hyperbola, whoſe vertex is S and centre L, and theſe two *loci* being deſcribed, their interſection will determine the point H, and then the ſides of the triangle become known.

Meſſrs. Hopper *and* Johnson, *alſo sent Solutions to this Queſtion.*

XIX. QUESTION 49, *by* ANALYTICUS.

Required the area of a curve whoſe equation is

$$y\sqrt{(1+ax^2+x^4)} = \frac{1-x^2}{1+x^2}.$$

SOLUTION, *by the Propoſer.*

From the given equation we have

$$y = \frac{1 - x^2}{1 + x^2} \times \frac{1}{\sqrt{(1 + ax^2 + x^4)}} = \frac{1 - x^2}{(1 + x^2)x} \times \frac{1}{\sqrt{(\frac{1}{x^2} + a + x^2)}}$$

and, therefore, $\dot{x}y$ or the fluxion of the area is

$\frac{(1 - x^2)\dot{x}}{(1 + x^2)x} \times \frac{1}{\sqrt{(\frac{1}{x^2} + a^2 + x^2)}}$. Now to transform this fluxion to one more convenient for integration, let us aſſume $z = \frac{1}{x} + x$, then $z^2 = \frac{1}{x^2} + 2 + x^2$ and $a - 2 + z^2 = \frac{1}{x^2} + a + x^2$. Alſo $\dot{z} = \dot{x} - \frac{\dot{x}}{x^2} = \frac{(x^2 - 1)\dot{x}}{x^2}$ and $\frac{\dot{z}}{z} = \frac{(x^2 - 1)\dot{x}}{(1 + x^2)x}$; therefore, by ſubſtitution, $\dot{x}y = \pm \frac{\dot{z}}{z\sqrt{(a - 2 + z^2)}}$, it being poſitive or negative according as x is greater or leſs than 1.

The fluent of this expreſſion may be found, as in Examp. 3, Art. 303, Simpſon's Fluxions, and when a is greater than 2, it is $=$

$\frac{1}{\sqrt{(a - 2)}} \times$ h. l. $\frac{\sqrt{(a - 2 + z^2)} - \sqrt{(a - 2)}}{z} = \frac{1}{\sqrt{(a - 2)}} \times$ h. l. $\frac{\sqrt{(\frac{1}{x^2} + a + x^2)} - \sqrt{(a - 2)}}{\frac{1}{x} + x}$. But when a is leſs than 2, the expreſſion $\frac{1}{\sqrt{(a - 2)}}$ will be impoſſible; and in this caſe the fluent will be expounded by

$\frac{1}{\sqrt{(a - 2)}} \times$ circ. arc, whoſe ſecant $= \frac{z}{\sqrt{(a - 2)}} \left(= \frac{1 + x^2}{x\sqrt{(a - 2)}}\right)$ to rad. unity.

XX. QUESTION 50, *by* AMICUS.

If a repreſent any arch of a circle, that arch is equal to the infinite product

sin. a × sec. $\frac{1}{2}a$ × sec. $\frac{1}{4}a$ × sec. $\frac{1}{8}a$ × sec. $\frac{1}{16}a$ × &c.

As this very curious expreſſion for a circular arch does not *ſeem generally* known, it is required to find its demonſtration.

SOLUTION,

SOLUTION, *by Mr.* JOHN DAWES, *Birmingham.*

It is well known that ſin. $2a = 2$ ſin. a cos. a, and if for a we write the ſucceſſive values $\frac{1}{2}a$, $\frac{1}{4}a$, $\frac{1}{8}a$, ... $\frac{1}{2^n}a$, we obtain

$$\begin{array}{ll} \text{1st} & \text{Sin. } a = 2 \text{ Sin. } \tfrac{1}{2}a \text{ Cos. } \tfrac{1}{2}a \\ \text{2d} & \text{Sin. } \tfrac{1}{2}a = 2 \text{ Sin. } \tfrac{1}{4}a \text{ Cos. } \tfrac{1}{4}a \\ \text{3d} & \text{Sin. } \tfrac{1}{4}a = 2 \text{ Sin. } \tfrac{1}{8}a \text{ Cos. } \tfrac{1}{8}a \\ \text{4th} & \text{Sin. } \tfrac{1}{8}a = 2 \text{ Sin. } \tfrac{1}{16}a \text{ Cos. } \tfrac{1}{16}a \end{array}$$

$$\text{And} \quad \text{Sin. } \frac{1}{2^n}a = \text{Sin. } \frac{1}{2^n}a \text{ Cos. } \frac{1}{2^n}a.$$

Now if the ſine $\frac{1}{2}a$ as found in the ſecond equation, be ſubſtituted for ſin. $\frac{1}{2}a$ in the firſt equation, we have ſin. $a = 4$ ſin. $\frac{1}{4}a$ cos. $\frac{1}{2}a$ cos. $\frac{1}{4}a$ and ſubſtituting for ſin. $\frac{1}{4}a$, its value found in the third equation, we have ſin. $a = 8$ ſin. $\frac{1}{8}a$ cos. $\frac{1}{2}a$ cos. $\frac{1}{4}a$ cos. $\frac{1}{8}a$, and by continuing this ſubſtitution we at laſt obtain

$$\text{ſin. } a = 2^n \text{ ſin. } \frac{1}{2^n}a \text{ cos. } \tfrac{1}{2}a \text{ cos. } \tfrac{1}{4}a \text{ cos. } \tfrac{1}{8}a \ldots\ldots \text{cos. } \frac{1}{2^n}a.$$

Now when n is indefinitely great, the arc $\frac{1}{2^n}a$ will be indefinitely ſmall, and therefore may be conſidered as equal to its ſine, and the cos. $\frac{1}{2^n}a$ will be equal to the radius (1); therefore, the above expreſſion becomes

$$\text{ſin. } a = a \text{ cos. } \tfrac{1}{2}a \text{ cos. } \tfrac{1}{4}a \text{ cos. } \tfrac{1}{8}a \text{ cos. } \tfrac{1}{16}a \text{ \&c.}$$

$$\text{or } a = \frac{\text{ſin } a}{\text{cos. } \frac{1}{2}a \text{ cos. } \frac{1}{4}a \text{ cos. } \frac{1}{8}a \text{ cos. } \frac{1}{16} \text{ \&c.}} =$$

$$\text{Sin. } a \times \frac{1}{\text{cos. } \frac{1}{2}a} \times \frac{1}{\text{cos. } \frac{1}{4}a} \times \frac{1}{\text{cos. } \frac{1}{8}a} \times \frac{1}{\text{cos. } \frac{1}{16}a}$$: but in any arch the ſecant is $= \frac{\text{rad.}^2}{\text{cos.}} = \frac{1}{\text{cos.}}$ therefore by ſubſtitution.

$$a = \text{ſin. } a \times \text{ſec. } \tfrac{1}{2}a \times \text{ſec. } \tfrac{1}{4}a \times \text{ſec. } \tfrac{1}{8}a \times \text{ſec. } \tfrac{1}{16}a, \text{ \&c.}$$

The inveſtigation of this expreſſion for a circular arc is given by Lacroix, in the Introduction to his Traité du Calcul. differentiel et intégral, page 79; but the invention belongs to Euler.

XXI. QUES-

XXI. QUESTION 51, *by* CURIOSUS.

In a work entitled "*Hemiſphærium Diſſectum*:" Authore *Ricardo Albio*, 4to. *Rome*, 1648, it is ſaid that the convex ſurface of a ſcalene cone is equal to a circle whoſe radius is a mean proportional between $\frac{1}{2}\sqrt{(h^2+r^2)}+\frac{1}{4}S+\frac{1}{4}s$ and r, where h is the cone's perpendicular height, r the radius of the baſe, S and s the greater and leſs ſlant heights. How is this rule derived?

SOLUTION, *by Mr.* LOWRY.

Let AIEZ (fig. 83, pl. 4.) repreſent the circular baſe of the cone, C its vertex, and AC, CE the greater and leſs ſlant ſides reſpectively; draw the diameter AOEG and demit the perpendicular CG, which will evidently repreſent the perpendicular altitude of the cone. Let I*i* be two points in the circumference of the baſe, indefinitely near to each other, join CI, C*i*, and draw the tangent IS meeting AG (produced if neceſſary) in S, and let CK be drawn at right angles to IS.

Then becauſe the arc I*i* is indefinitely ſmall, the triangle IC*i* will be equal to the arc I*i*, multiplied by half the perpendicular CK, or taking (on the diameter produced) AH, A*h* equal to the arcs EI, E*i* reſpectively, and the perpendicular HK = half CK, the triangle IC*i*, will be equal to the rectangle *h*HK; and as the ſurface of the cone is made up of an indefinite number of theſe little triangles IC*i*, or their equals the rectangles *h*HK, it follows that if AF be taken equal to the ſemi-circumference AIE, and ordinates be taken at every point H correſponding to every point I round the circumference, equal to half the perpendicular CK drawn to the tangent at that point, the area of the curve AIE will be equal to half the ſurface of the cone. To determine the exact area of this curve, is however no eaſy taſk, nor would it be of any uſe in the inveſtigation of the rule mentioned in the queſtion, which is only an approximation, and ſeems to have been derived from ſome ſuch method as follows:

When the points AI coincide, it is evident that CA and CK will alſo coincide, and therefore FT the ordinate correſponding to the point F (or A) will be equal to half CA = $\frac{1}{2}$ S, and in the ſame way it appears that AL the ordinate correſponding to the point A (or E) is equal to half CE = $\frac{1}{2}s$. Let AF be biſected at H′ *and draw the ordinate* H′ K′, then H′ is the point correſponding to I the middle of the arc AIE, and in this caſe the tangent IS will be parallel to the diameter AE, and the line joining GK will be equal and parallel to IO the radius of the cone's baſe; therefore CK will be $= \surd(GC^2 - GK^2)$ and $H'K' = \frac{1}{2}\surd(CG^2 - GK^2) = \frac{1}{2}\sqrt{h^2 + r^2}$. Now it is very evident that the area of the curve will be nearly equal to the ſum of the two trapezoids TFH'K' and H'K'EL, the ſmall defect in the one being nearly compenſated by the exceſs in the other, therefore the area may be conſidered as equal to $\left(\frac{FT}{2} + H'K' + \frac{AL}{2}\right)\frac{FA}{2} = \left(\frac{1}{2}\sqrt{h^2 + r^2} + \frac{S}{4} + \frac{s}{4}\right) \times \frac{FA}{2}$, and therefore the whole convex ſurface of the cone will be equal to twice that ſpace $= \left(\frac{1}{2}\surd(h^2 + r^2) + \frac{S}{4} + \frac{s}{4}\right) \times r \times 3{\cdot}14159$ &c. (FA being equal to $r \times 3{\cdot}14159$) which is evidently equal to a circle whoſe radius is a mean proportional between $\frac{1}{2}\surd(h^2 + r^2) + \frac{S}{4} + \frac{s}{4}$ and r.

This Rule will be ſufficiently exact for practical purpoſes, eſpecially when the ſlant ſides are nearly equal; and in caſes where greater accuracy is neceſſary, the area may be approximated to any degree of exactneſs, and with the greateſt facility, by the method of equidiſtant ordinates.

XXII. QUESTION 52, *by* ALIQUIS.

Having given the baſe of a triangular pyramid, and the three plane angles at the vertex; to find the other dimenſions?

SOLUTION, *by Mr.* LOWRY.

To render the Solution of this Problem as ſimple as poſſible, let us conceive the ſurface of the pyramid to be expanded on a plane as repreſented in fig. 84, pl. 4; then we ſhall have given the angles BAC, CAD, DAE, and the oppoſite ſides BC, CD, DE, to find the other ſides AB, AC, AD and AE, whereof AB, and AE are equal. Now aſſume AF = AI of any given length, and draw FG, GH, HI parallel to BC, CD and DE reſpectively, and drop the perpendiculars FL, IK; then the figure FGHIA will be ſimilar to BCDEA, and therefore the lines FG, GH, HI will have the ſame ratio as the given lines BC, CD, DE; and becauſe AF $=$ AI is given, the lines FL, AL, IK and AK will alſo be given. Now put FL $= a$, AL $= b$, IK $= c$, AK $= d$, coſ. $\angle$ EDC $= v$, LG $= x$, KH $= y$ and let m, n, r repreſent the ratios of the ſquares of FG, GH and HI reſpectively; then $FG^2 = a^2 + x^2$ and $IH^2 = c^2 + y^2$ therefore, $a^2 + x^2 : c^2 + y^2 :: m : r$

$$\text{or } x^2 = \frac{m}{r} y^2 + \frac{m}{r} c^2 - a^2.$$

Again, by trigonometry, $GH^2 = GA^2 + HA^2 - 2v.GA.AH$, and if for AG, and AH, there be written $b + x$, and $d + y$, and w^2 be put for $b^2 + d^2 - 2vbd$, we have $GH^2 = x^2 + y^2 + 2(b - vd - vy)\,x + 2(d - vb)\,y + w^2$ and, therefore, $x^2 + y^2 + 2(b - vd - vy)x + 2(d - vb)\,y + w^2 : c^2 + y^2 :: n : r$, and this by reducing becomes $2(b - vd - vy)\,x = \frac{n}{r} c^2 + \frac{n - r}{r}.$ $y^2 - x^2 - 2(d - vb)\,y - w^2$, and ſubſtituting for x its value as found above, we have,

$$2(b\text{-}vd\text{-}vy).\left(\frac{m}{r}y^2 + \frac{m}{r}c^2 \text{-} a^2\right)^{\frac{1}{2}} = \frac{n-m}{r}c^2 + \left(\frac{n-r-m}{r}\right)y^2 \text{-} 2(d\text{-}vb)y \text{-} w^2 + a^2$$

which being expanded and properly reduced, we may obtain the value of y by a biquadrate, when the given quantities are expreſſed in numbers, and from thence the other parts will readily become known.

It is clear from the above, that the problem cannot in general be conſtructed by plane Geometry, but it may be eaſily done by the conic ſections; the locus expreſſed by each of the above equations being an hyperbola.

Or if it ſhould be more deſireable to have the reſulting equation expreſſed in terms of the quantities given in the queſtion, without firſt finding the dimenſions of a ſimilar pyramid, we may proceed as follows: Put x, y, and z, for the three ſlant ſides, m, n and s for the coſines of the given angles, and a, b, c, for the ſides of the baſe oppoſite to thoſe angles; then by trigonometry we have the following equations,

$$x^2 + y^2 - 2xym = a^2$$
$$y^2 + z^2 - 2yzn = b^2$$
$$x^2 + z^2 - 2xzs = c^2.$$

Now let us aſſume $y = wx$ and $z = vx$, then our equations become

$$x^2\,(1 + w^2 - 2wm) = a^2$$
$$x^2\,(w^2 + v^2 - 2wvn) = b^2$$
$$x^2\,(1 + v^2 - 2vs) \;\; = c^2.$$

And from hence we have three values of x^2 which being equated we obtain

$$a^2\,(w^2 + v^2 - 2wvn) = b^2\,(1 + w^2 - 2wm)$$
$$a^2\,(1 + v^2 - 2vs) \;\; = c^2\,(1 + w^2 - 2wm)$$

or,

$$a^2v^2 - 2wna^2v - b^2(1 + w^2 - 2wm) - a^2w^2 = 0$$
$$a^2v^2 - 2sa^2v - c^2(1 + w^2 - 2wm) - a^2 = 0.$$

Now to exterminate one of the unknown quantities, as v, put $= 2wna^2$, $Q = b^2(1 + w^2 - 2wm) - a^2w^2$, $R = 2sa^2$ nd $S = c^2(1 + w^2 - 2wm) - a^2$; then we have

$$a^2v^2 - Pv - Q = 0$$
$$a^2v^2 - Rv - S = 0.$$

Subtract the ſecond equation from the firſt, and there reſults

$$(R - P)v + S - Q = 0; \text{ or, } v = \frac{Q - S}{R - P}.$$

Again ſubtract the ſecond equation multiplied by Q, from the irſt equation multiplied by S, and divide the remainder by v, nd we have

$$^2(Q - S)v - PS + QR = 0; \text{ or, } v = \frac{PS - QR}{a^2(Q - S)},$$

nd equating theſe values of v, we have

$^2(Q - S)^2 = (PS - QR) \cdot (R - P)$; ~~which, by~~ ſubſtituting or P, Q, R and S, their reſpective values become

$$(b^2 - c^2)(1 + w^2 - 2wm) + 1 - w^2]^2 = [(wnb^2 - sc^2)(1 + w^2 - 2wm) + (sw^2 - wn)a^2] \cdot 4(s - wn)a^2.$$

Where the value of w may be found by an equation of the 4th ower, and from thence n and x (and of courſe y and z) become nown.

XXIII. QUESTION 53, *by Mr.* WM. WALLACE, *of the Royal Military College.*

Let AB, AC be two ſtraight lines given by poſition, and let DE be a line of a given length terminated by them at D and E, let DV, EV, be perpendiculars to the lines AB, AC, meeting in V. It is required to determine the *Locus* of the point V.

FIRST SOLUTION, *by Miſs* MARIA PORTER, *London.*

Suppoſe that AB, AC, (fig. 85, pl. 4.) are the two lines placed in their given poſition, DE the given line drawn as per queſtion, and DV, EV the two perpendiculars. Then, ſince the ∠s ADV, AEV are right ∠s, the ∠ A + ∠ DVE muſt be equal to two right angles; but the ∠ A is given, therefore, the ∠ DVE is alſo given, it being equal to the ſupplement of the ∠ A.

Take PQ = the given line DE and (Euc. 33. b. iii.) defcribe fegment of a circle P*m*Q to contain an angle equal to the give angle DVE; draw P*m* to make the angle QP*m* = DEV; ar join Q*m*. Let *o* be the centre of the circle, draw the diamet *m o n* and join P*n*, Q*n*; and draw AV: then, becaufe the ang DEV = *m* PQ, and ∠ EVD = P*m*Q, therefore, the triangl P*m*Q, and EVD are equiangular; therefore, becaufe PQ = D the two triangles are equal in every refpect, and P*m* = EV ar *m*Q = VD. And becaufe the two right angles *m*P*n* and VE are equal and the two parts *m*PQ and VED likewife equal, the r mainders AED and *n*PQ are equal: Hence it is manifeft that th two triangles AED and *n*PQ are equiangular, therefore, they a equal in all refpects, becaufe ED = PQ. And becaufe EV = P*m* and EA = P*n* and the angles at E and P are right angle therefore, AV is equal to *m n*, that is to the diameter of a give circle: therefore, the locus of the point V is a given circle which A is the centre.

Second Solution, *by Mr.* Thos. Myers, *Hammerſmith*

Fig. 85, pl. 4. The angles at D and E being right angle the points A, D, V, E, are in the circumference of a circl whofe diameter is AV, and if the diameter DF be drawn meetin the circle again in F, and EF be joined, the angle DFE is equal to the given angle DAC, and DEF is a right angle, and DE is given in magnitude; therefore, DF and its equal AV will be given, and, confequently, the locus of the point V will be a circle whofe centre is A and radius AV.

Third Solution, *by* Porus.

Draw EG (fig. 86, pl. 4,) perpendicular to DE and VG perpendicular to VD meeting EG in G; join DG, AV, and taking AC = AV let fall CB perpendicular to AB. Then will V, G, E, D, be points in a circle whofe diameter is DG, and A, D, V, E points in a circle whofe diameter is AV. Confequently, the angles VEG, VDG, and AVD, AED are equal, but VEG is equal to AED, becaufe GED, VEA are equal by conftrction; therefore, in the triangles DVG, DAV, the angles VDG AVD and GVD, ADV are equal, and the fide DV is common to both; hence DG is equal to AV. Now the angle EVD or its equal BAC is equal to EGD, the angle ABC equal to DEG, and AC equal to AV or DG by the conftruction; confequently, BC is equal to DE, but DE and the angles BAC, ABC are conftant by the hypotheſis and conſtruction, wherefore, AC or its equal AV is conſtant, and the locus of V is a circle.

Cor. It is evident that if DV, VE had been drawn ſo as to make the angles ADV, CEV equal, the locus of the point of concourſe V would have alſo been a circle, for if the angle DEG be made equal to AEV, the angle DVG equal to CEG, and ABC equal AEV, the ſteps of the demonſtration will be exactly the ſame.

Solutions were alſo received from Meſſrs. Bazley, Dawes, Hopper, Johnſon, Lowry, *and* Smith.

XXIV. QUESTION 54, *by* URSA MINOR.

Three ſtraight lines being given by poſition, and three points being alſo given, it is required to deſcribe a rectilineal figure, whoſe ſides may paſs through the given points, and whoſe angles may be upon the lines given by poſition.

SOLUTION, *by Mr.* LOWRY, *R. M. College.*

The Analyſis which I have given to the 42d Queſtion, will aſſiſt us in reſolving this problem, and alſo in deſcribing a rectilineal figure of any number of ſides whatever, ſo that its angular points may fall on ſtraight lines given by poſition, and its ſides paſs through given points.

Let AB, BC, AC, (fig. 87, pl. 4.) be the three ſtraight lines given by poſition; D, E, F, the given points, and IGK the triangle to be deſcribed, ſo as to have its angular points on the ſtraight lines given by poſition, and its ſides paſſing through the given points. Now proceeding exactly as in that Analyſis, (ſuppoſing the points H, C to coincide,) we may find a point R, and the poſition of the ſtraight line RS, ſuch, that drawing KG, IG, through the points E, F to interſect in AB, and meet RS in S, and AC in I, KS may be equal to IC. And in the ſame manner we may find a point L, and the poſition of the ſtraight line LM, ſuch, that drawing GK, ID through the points E, D to interſect in BC, and meet RS in S, and LM in M, RS may be equal to LM; then it is obvious that the lines RS, LM, and IC will be equal, when the triangle is deſcribed in the manner required, and, therefore, we have only to draw IK through the point D, to meet AC in I, and LM in M, ſo that IC may be equal to LM, which is a particular caſe of the *Sectio Rationis* of Apollonius.

XXV. QUESTION 55, *by* QUINBUS FLESTRIN,

Given the baſe and the length of a line from the vertical angle to a given point therein, to conſtruct the triangle when the vertical angle is a maximum.

FIRST SOLUTION, *by Mr.* THOMAS HOPPER.

Let CB (fig. 91, pl. 4.) be the given baſe, and P the given point; with the centre P and diſtance equal to the given line deſcribe a ſemicircle AFD meeting the baſe produced in A and D; take DE, ſuch, that DE : CD :: BD : AB — CD, and from E draw EF to touch the ſemicircle at F; join BF, CF and BFC will be the triangle required.

Through the points BFC deſcribe a circle; then becauſe DE : CD :: BD : AB — CD, we have by compoſition DE : CE :: BD : AC, and again, by alternation and compoſition, DE : BC :: CE : AE; therefore, $BC \cdot CE = DE \cdot AE = EF^2$, and, conſequently, EF is a tangent to both the circles, (Eu. 37. b.iii.) and, therefore, they touch each other at F, and the angle BFC will in that caſe be a maximum, as is well known.

SECOND SOLUTION, *by Mr.* COLLINS, *Kenſington.*

Let AB (fig. 92, pl. 4.) be the given baſe, D the given point therein, with which as a centre and radius = the given line, drawn from the vertex to the given point, deſcribe the circle *a*C*b*, then by Prop. 45, Simpſon's Geometry, deſcribe a circle to paſs through A and B, and touch the other circle in C, join AC and BC, and ABC will be the triangle required.

DEMONSTRATION. By the conſtruction the baſe AB, the ſegments AD, DB, and the line DC are all equal to the given ones. We have therefore only to prove that the angle ACB is a maximum. To do this, let any other lines be drawn to the circle as at *c*, and the angle ACB will be equal to the angle A*c*B, and, conſequently, greater than the angle A*d*B formed by any other two lines A*d*, B*d*, drawn to any other point in the circle *a*C*b*.

Third Solution, *by Mr.* Thomas Bazley.

Analysis. Suppoſe it done, ABC, (fig. 93, pl. 4.) the required Δ, S the given point in the baſe, SC the given right line, and ACBK the circumſcribing circle whoſe centre is O. Then, becauſe the ∠ C is a maximum, the circle deſcribed from centre S with radius SC muſt touch the circumſcribing circle in C. For if C be ſuppoſed to fall in any other point of the exterior circle than the point of contact, it is plain then, that the ∠ C would be leſs than the ∠ in the circumſcribing circle, that is, leſs than that at the point of contact. Now it is known that C, O, S are in the ſame right line: produce it to meet the circumſcribing circle in R and drop OD perpendicular to AB. Then $AD = \frac{1}{2}AB$ is given, and ſince the point S is given, therefore SD is given; conſequently, $AD^2 - DS^2 = AO^2 - OS^2 = CS \times SR$ is given. But CS is given, therefore SR is given, and, conſequently, $\frac{CS - SR}{2} = OS$ is given, and the conſtruction is very evident.

Limitation. It is requiſite that SC be *greater* than either of the ſegments SA, SB, when the vertical ∠ is required to be a maximum; and *leſs* when a minimum.

Ingenious Solutions were likewiſe received from Meſſrs. Johnſon, Lowry, Merones Minor, Myers, *and* Pickering.

XXVI. QUESTION 56, *by Mr.* James Cunliffe.

Divide the ſum of three ſquare numbers in arithmetical proportion into three other ſquare numbers in arithmetical proportion, and give an example in whole numbers as ſmall as may be.

Solution, *by Mr.* James Ivory, *R. M. College.*

When three numbers are in arithmetical progreſſion, it is eaſy to prove that the ſum of all the three is triple of the middle number: therefore, if two ſets of three numbers in arithmetical progreſſion have the ſame ſum, it is obvious that the middle term of each ſet muſt be the ſame; and, on the contrary, if the middle terms be the ſame, the ſums of the two ſets will likewiſe be the ſame.

Let then a^2, b^2, c^2 and a'^2, b^2, c'^2 be two ſets of ſquare numbers in arithmetical progreſſion, having the ſame middle term b^2, and, conſequently, the ſame ſum $3b^2$: then

$$2b^2 = a^2 + c^2 = a'^2 + c'^2$$

or

$$b^2 = \left(\frac{a+c}{2}\right)^2 + \left(\frac{a-c}{2}\right)^2 = \left(\frac{a'+c'}{2}\right)^2 + \left(\frac{a'-c'}{2}\right)^2.$$

It is manifeſt, therefore, that the Queſtion is reduced to the finding of a ſquare number that may be reſolved into the ſum of two ſquares twice. Attending to the principles laid down in the Solution of Queſtion 23d, No. II. of the Repoſitory, it will appear that the number ſought muſt contain prime diviſors of the form $4x + 1$; and, as a leaſt value is wanted, we may neglect all the other diviſors which would only increaſe the number ſought without rendering it more adapted to the Solution of the Queſtion. We may infer likewiſe from the ſame principles, that no ſingle prime number of the form $4x + 1$ will ſatisfy the required conditions; becauſe a prime number of the form $4x + 1$, as well as its ſquare, can be the ſum of two ſquares only once. Taking then a number compoſed of two ſuch prime factors, we ſhall readily find that the number itſelf is the ſum of two ſquares twice, and that its ſquare is the ſum of two ſquares no leſs than four times. Thus if $p = \alpha^2 + \beta^2$ and $p' = \alpha'^2 + \beta'^2$: then

$$pp' = \begin{cases} (\alpha\alpha' + \beta\beta')^2 + (\alpha\beta' - \alpha'\beta)^2 = A^2 + B^2 \\ (\alpha\alpha' - \beta\beta')^2 + (\alpha\beta' + \alpha'\beta)^2 = C^2 + D^2, \end{cases}$$

And

$$p^2\,p'^2 = \begin{cases} (A^2 - B^2)^2 + 4A^2\,B^2 \\ (C^2 - D^2)^2 + 4C^2\,D^2 \\ A^2\,p'^2 + B^2\,p'^2 \\ A'^2\,p^2 + B'^2\,p^2 \end{cases}$$

Hence it is evident that the number ſought muſt be the product of two primes of the form $4x + 1$; and that each ſuch number will furniſh no leſs than 6 Solutions of the Queſtion, *viz.* as many Solutions as there are combinations of two things in four.

The two leaſt primes of the form $4x + 1$ are 5 and 13; their product is 65: now

$$\begin{aligned} 65^2 &= 33^2 + 56^2 \\ 65^2 &= 63^2 + 16^2 \\ 65^2 &= 52^2 + 39^2 \\ 65^2 &= 25^2 + 60^2. \end{aligned}$$

From theſe we derive the following ſets of three ſquares in *arithmetical progreſſion*,

$$\begin{array}{lll} 23^2, & 65^2, & 89^2 \\ 47^2, & 65^2, & 79^2 \\ 13^2, & 65^2, & 91^2 \\ 35^2, & 65^2, & 85^2. \end{array}$$

The ſum of each ſet being the ſame and equal to 3×65^2; and theſe ſets of numbers furniſh 6 different Solutions of the Queſtion, by taking all their combinations, two and two.

But if we ſuppoſe $p = p'$, then the four forms for $p^2 p'^2$ will give only two forms for p^4 (vide Queſt. 23, N. 2.): and thus the ſquare of every prime of the form $4x + 1$ will give an example, admitting only one Solution of the Queſtion. The leaſt prime of the form $4x + 1$ is 5: its ſquare is 25: and we have

$$25^2 = 7^2 + 24^2$$
$$25^2 = 15^2 + 20^2$$

whence we have

$$\begin{array}{lll} 31^2, & 25^2, & 17^2 \\ 35^2, & 25^2, & 5^2 \end{array}$$

which are the leaſt numbers that ſatisfy the Queſtion.

XXVII. QUESTION 57, *by Mr.* CUNLIFFE.

Let a ſlender heavy inflexible rod AB, loaded with a weight, be placed with its upper end B leaning againſt an inclined plane DE, and its lower end A upon the curve DA, ſituated in a vertical plane at right angles to the inclined plane. It is required to inveſtigate the nature of the curve DA, ſo that the rod may reſt in equilibrio when placed in any inclination.

SOLUTION, *by Mr.* LOWRY, *R. M. College.*

Let *d*D (fig. 94, pl. 4.) be the given plane making any given angle with the horizontal plane IDK, DAQ a portion of the required curve, AB the given rod with the weight attached reſting in equilibrio in any aſſumed poſition, and E their centre of gravity, then becauſe AB is of a given length the parts AE, BE, will alſo be given.

Now by a well known principle in mechanics, when the rod reſts in equilibrio, the centre of gravity will be at the loweſt place, or the neareſt to the horizontal plane IK that it can poſſibly get; and as the rod is to reſt in equilibrio in whatever poſition it is placed, it is manifeſt that the centre of gravity muſt in every poſition be equally diſtant from the plane IK, for if it was not the rod would not be at reſt, but would ſlide along the curve till it came to that poſition where the centre was at the loweſt. The *locus* of the centre of gravity will, therefore, be the straight line QEC, drawn parallel to IDK, C being the centre of gravity of the rod in its initial poſition, and a given point, becauſe CD is = AE, and, therefore, CEQ is given in poſition.

Now to inveſtigate the nature of the curve, draw AS parallel to QC, and put $a =$ AE, $b =$ AB, coſine of the angle ASB $= c$, CS $= x$, and AS $= y$; then, by ſimilar triangles, $a : x :: b : \frac{bx}{a} =$ BS, and, by trigonometry, $y^2 + \frac{b^2}{a^2} x^2 + 2\frac{cb}{a} xy = b^2$, which is manifeſtly an equation to a conic ſection. And becauſe the coſine of an arc is always leſs than the radius $\frac{c^2 b^2}{a^2}$, or the ſquare of half the co-efficient of xy is leſs than $\frac{b^2}{a^2}$ the co-efficient of x^2; and the ſquares x^2, y^2 have the ſame ſign; therefore, the curve is an ellipſe, whoſe centre is C. Maclaurin's Algebra, Part iii. Section 29.

XXVIII. QUESTION 58, *by Mr.* BAZLEY

AC, CB are two rods or right lines connected at C by a moveable joint, and whilſt the end C of the rod BC revolves round the fixed point B let the point A of the rod AC be carried along the right line AB paſſing through B; it is required to determine the equation and area of the curve deſcribed by P, a given point in AC.

SOLUTION, *by Mr.* LOWRY.

With the centre B (fig. 95, pl. 4.) and radius BC deſcribe the circle bCH, then becauſe the rod BC is of a given length, and revolves round the fixed pole B, the end C will move round in the circumference bCH, while the point A moves along the ſtraight line aH. Let ab be taken equal to AC; and $ap =$ Hp' $=$ AP; then when C is at b or the rods are extended ſo as to coincide with aB, the point P will coincide with p, and when C comes to H, then P and p' will coincide; therefore, pp' are the points where the curve meets the axis AH. On AB drop the perpendiculars PQ, CE, and put AC $= a$, AP $= b$, PC $= c$, BC $= r$, $v = 3{\cdot}14159$ &c. pQ $= x$, and PQ $= y$;

then, by ſimilar triangles, AP : PQ :: AC : CE $= \frac{a}{b} y$, and, therefore, $AE^2 = a^2 - \frac{a^2}{b^2}y^2$. Again $a^2 : c^2 :: AE^2 : QE^2 = c^2 - \frac{c^2}{b^2}y^2$ or, $QE = \frac{b}{c}\sqrt{(b^2 - y^2)}$; and $BE^2 = BC^2 - CE^2 = r^2 - \frac{a^2}{b^2}y^2$ or, $BE = \frac{b}{a}\sqrt{\left(\frac{r^2 b^2}{a^2} - y^2\right)}$. Now $Bp = r + c$, and, therefore, pQ, or $x = r + c - BE - QE = r + c - \frac{b}{a}\sqrt{\left(\frac{r^2 b^2}{a^2} - y^2\right)} - \frac{b}{c}\sqrt{(b^2 - y^2)}$, which is the equation of the curve; and the fluxion of the area $\dot{x}y$ is

$$= \frac{\frac{a}{b}y^2\dot{y}}{\sqrt{\left(\frac{r^2 b^2}{a^2} - y^2\right)}} + \frac{\frac{c}{b}y^2\dot{y}}{\sqrt{(b^2 - y^2)}}$$ the fluent of which, by Article 279 of Simpſon's Fluxions, is

$$= \frac{r^2 b}{2a}A - \frac{ay}{2c}\sqrt{\left(\frac{r^2 b^2}{a^2} - y^2\right)} + \frac{bc}{2}B - \frac{c}{2b}y\sqrt{(b^2 - y^2)},$$ where A and B are the circular arcs, whoſe ſines are $\frac{ay}{rb}$, and $\frac{y}{b}$ the radius being unity. Now when x becomes $=$ to pp', y is $= o$, and in this caſe the arcs A and B are each equal to half the circumference, and hence the whole area is $= \frac{vb}{2a}(r^2 + ac.)$

It is obvious from the manner of deſcribing the curve, that it is of an oval form, having the parts on the different sides of pp' exactly equal and ſimilar, and is a line of the 4th order in all caſes except when the rods are of equal length, and then it becomes an ellipſis, whoſe centre is B; for if a be wrote for r in the general equation of the curve, we obtain after proper reduction,

$$2(a+c)\,x - x^2 = \left(\frac{c+a}{b}\right)^2 y^2,$$ which is an equation to an ellipſe whoſe ſemi-axes are $c + a$ and b.

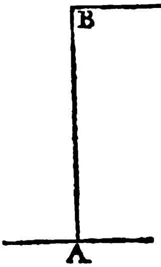
When P has deſcribed a quadrant of this ellipſe, BC will be perpendicular to the diameter lH, and the rods will coincide, so that during the remaining part of the revolution, P will deſcribe the quadrant of a circle concentric to the given one; unleſs we ſuppoſe the rod AC to make a tranſit at B, and ſlide paſt the other rod, ſo that A may continue to move from B towards H, and then P will deſcribe the remaining part of the semi-ellipſe.

Moreover, if the rod BC, be longer than AC, it is obvious that BC can make only a partial revolution round B, ſo that A may keep in the line aH. for when AC becomes perpendicular to that line, it is manifeſt that if BC was moved any nearer to H, the point A would neceſſarily quit *a*H, and move in some other line nearer to the circle, and, conſequently, when BC has arrived at that poſition where AC is perpendicular to *a*H, it muſt move back again till it coincides with B*b*; and then if *bp*′ be taken = PC, P will coincide with *p*′, and *pp*″ will be the axis of the curve.

Meſſrs. Dawes, *and* Bazley *the propoſer, alſo anſwered this queſtion.*

XXIX. QUESTION 59, *by Sir* TITUS TRIPLICATE.

Suppoſe a given cylindrical tube or pipe ABC, whereof AB is perpendicular, and BC parallel to the horizon, to revolve round AB as an axis with a given angular velocity; the lower end A ſtanding in water, whoſe ſurface is at A, and the whole tube being previouſly filled; it is required to determine the velocity with which the water will iſſue from the extremity C.



SOLUTION, *by the Propoſer.*

Put
a = length of the arm BC in feet
b = height of AB in feet
t = time of a revolution in ſeconds
g = $16\frac{1}{12}$ feet, the ſpace deſcended by heavy bodies in 1″
p = 3·1416 = periphery of a circle whoſe diam. is 1.

If we ſuppoſe the centrifugal force to be as the velocity generated thereby in 1″; then by the laws of central forces we ſhall have the centrifugal force of a particle $\dot{x}$ at x diſtance from B, equal to $\left(\frac{2p'}{t}\right)^2\dot{x} \div x = \frac{4p^2x\dot{x}}{t^2}$, and the fluent $\frac{2p^2a^2}{t^2}$ is the centrifugal motive force of the column x; whence, when $x = a$, the whole centrifugal motive force of the water in the arm BC is $\frac{2p^2a^2}{t^2}$. But by the laws of gravity $2ga : a :: \frac{2p^2a^2}{t^2} : \frac{p^2a^2}{gt^2}$ = the length of the column whoſe preſſure is equal to the centrifugal force.

Now it is plain from the nature of the queſtion that the column *impelling the* water out of the extremity of the arm BC is $\frac{p^2a^2}{gt^2}$

— b

$-b$ and that the column moved is $a+b$ which alſo expreſſes the ſpace through which the water is accelerated. Hence the circumſtances of the problem now appear to be the ſame as if a conſtant head of water of the height $\frac{p^2a^2}{gt^2}-b$, impelled a column of water horizontally at the bottom of the length $a+b$. Therefore, the accelerative force is $\left(\frac{p^2a^2}{gt^2}-b\right)\div(a+b)$, and the ſpace or length of acceleration $=a+b$; and, conſequently, the velocity generated $=(v=2\sqrt{fgs})\ 2g^{\frac{1}{2}}\sqrt{\left(\frac{p^2a^2}{gt^2}-b\right)}$; or to adapt it to practical purpoſes, we may ſubſtitute for $2g^{\frac{1}{2}}$ the quantity $5{\cdot}3$ as determined by the Experiments of l'Abbé Boſſuet and others; and then the velocity is $5{\cdot}3\sqrt{\left(\frac{p^2a^2}{gt^2}-b\right)}$.

It may alſo be added, that b muſt not exceed 33 feet, becauſe a greater column cannot be ſupported by the atmoſphere's preſſure.

Solutions were alſo received from Meſſrs. Collins, Myers, *and* Pickering.

XXX. PRIZE QUESTION 60, *by Mr.* JAS. IVORY.

THEOREM. Let n be any integer number, and let m be another integer number leſs than n: then

$$0=n^m-n\cdot(n-1)^m+n\cdot\frac{n-1}{2}\cdot(n-2)^m-n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot(n-3)^m+$$

&c. the co-efficients being the ſame with the co-efficients of the binomial quantity $(a-b)^n$, and the progreſſion, n^m, $(n-1)^m$, $(n-2)^m$ to be continued till we arrive at 1^m. Required the inveſtigation.

FIRST SOLUTION, *by the Propoſer, Mr.* IVORY.

Let $A+Bx+Cx^2+Dx^3+\ldots\ldots\ldots\ldots Mx^m$ be an integral function of x, in which x^m is the higheſt power of x: Suppoſe the ſeveral terms of a ſeries in arithmetical progreſſion to be ſubſtituted for x, and denote the reſulting values of the function by

$$y',\ y'',\ y''',\ y^{iv},\ \&c.$$

Then, according to what is demonſtrated by all the writers on finite differences, the ſeveral differences of the mth order in the progreſſion y', y'', y''', y^{iv}, &c. will be all equal to one another; and of conſequence the differences higher than the mth order will be equal to nothing. Now the general expreſſion for the firſt of the nth order of differences is

$$y' - ny'' + n,\ \frac{n-1}{2}.\ y''' - n.\ \frac{n-1}{2}.\ \frac{n-2}{3}.\ y^{iv} + \&c.$$

which expreſſion will be equal to nothing when m is leſs than n, according to what has juſt been ſhewn. It is evident that the Queſtion is only a very particular caſe of the propoſition now demonſtrated.

The Queſtion may likewiſe be demonſtrated in the following manner. Let e denote the number of which the hyperbolic logarithm is unity; then

$$e^{x} = 1 + x + \frac{x^{2}}{1\cdot 2} + \frac{x^{3}}{1\cdot 2\cdot 3} + \frac{x^{4}}{1\cdot 2\cdot 3\cdot 4} + \&c,$$

$$\text{therefore, } e^{x} - 1 = x + \frac{x^{2}}{1\cdot 2} + \frac{x^{3}}{1\cdot 2\cdot 3} + \&c:$$

For the ſake of brevity write S for the ſeries on the right hand of the equation and raiſe both ſides to the nth power; then

$$S^{n} = e^{nx} - n\cdot e^{(n-1)x} + n\cdot\frac{n-1}{2}.e^{(n-2)x} - n\cdot\frac{n-1}{2}.\frac{n-2}{3}.e^{(n-3)x} + \&c.$$

develope each term of the expanded binomial into a ſeries proceeding by the powers of x, and having collected all the co-efficients of the ſame power of x into one ſum, the equation may be thus written,

$$S^{n} = A^{(0)} + A^{(1)} \times x + A^{(2)} \times x^{2} + A^{(3)} \times x^{3} \ldots . A^{(m)} \times x^{m}$$

where

$$A^{(0)} = 1 - n + n.\frac{n-1}{2} - n.\frac{n-1}{2}.\frac{n-2}{3} + \&c. = 0,$$

$$A^{(1)} = n - n \times (n-1) + n.\frac{n-1}{2} \times (n-2) - \&c.$$

$$A^{(2)} = \tfrac{1}{2} \times \Big\{ n^{2} - n \times (n-1)^{2} + n.\frac{n-2}{2} \times (n-2)^{2} - \&c.$$

$$A^{(3)} = \frac{1}{1\cdot 2\cdot 3} \times \Big\{ n^{3} - n \times (n-1)^{3} + n.\frac{n-1}{2} \times (n-2)^{3} - \&c.$$

And in general,

$$A^{(m)} = \frac{1}{1.2.3.4\ldots\ldots m} \times \left\{ n^m - n \times (n-2)^m + n.\frac{n-1}{2} \times (n-3)^m - \&c. \right.$$

A^m denoting the co-efficient of x^m whatever number m may repreſent. Now the equation that has been found is independent of the value of x, which could not be the caſe unleſs the two ſeries were identical, and the co-efficient of any power of x in the one ſeries equal to the co-efficient of the ſame power of x in the other ſeries. But all the powers of x, lower than the nth are evidently wanting in the ſeries S^n: therefore, the co-efficients of the ſame powers of x muſt be all evaneſcent in the other ſeries, that is, we muſt have $A^{(m)} = 0$, ſo long as m is leſs than n; or when m is leſs than n,

$$n^m - n\,(n-1)^m + n.\frac{n-1}{2}\,(n-2)^m - \&c. = 0.$$

Cor. 1. The loweſt power of x in the ſeries S^n is x^n and its co-efficient is 1: therefore, we muſt likewiſe have $A^{(n)} = 1$, that is

$$1 = \frac{1}{1.2.3.4\ldots.n} \times \left\{ n^n - n(n-1)^n + n.\frac{n-1}{2}(n-2)^n - \&c. \right\}$$

$$\text{or } \left\{ n^n - n(n-1)^n + n.\frac{n-1}{2}(n-2)^n - \&c. \right\} = 1.2.3.4\ldots\ldots n.$$

Cor. 2. Let $O = 1 + \frac{x}{2} + \frac{x^2}{1.2.3} + \frac{x^3}{1.2.3.4} + \&c.$ then the co-efficient of x^m in the ſeries O^n will be

$$\frac{1}{1.2.3.4\,..(n+m)} \times \left\{ n^{n+m} - n(n-1)^{n+m} + n.\frac{n-1}{2}(n-2)^{n+m} - \&c. \right.$$

Second Solution, *by Mr.* Bazley, *Bolton.*

If we take any reſidual $(z-1)$ whoſe firſt term' is any variable quantity, and the ſecond term unity, and involve it by the binomial theorem, we have, as is well known,

$$z^n - nz^{n-1} + n.\frac{n-1}{2}.\,z^{n-2} - \&c. = (z-1)^n;$$

but in what follows it will be convenient, for brevity's ſake, to put $A = n$, $B = n.\frac{n-1}{2}$, $C = n.\frac{n-1}{2}.\frac{n-2}{3}$, &c.

ſo that

$$z^{n} - Az^{n-1} + Bz^{n-2} - Cz^{n-3} \&c. = (z-1)^{n}.$$

Take fluxions in this expreſſion and divide by $\dot{z}$, then

$$nz^{n-1} - (n-1)Az^{n-2} + B(n-2)z^{n-3} - \&c. = n(z-1)^{n-1} \ldots\ldots (1)$$

which when $z = 1$, or $z - 1 = 0$ gives

$$n - A(n-1) + B(n-2) - C(n-3) + \&c. = 0.$$

Multiply the expreſſion (1) by z, take fluxions and divide by $\dot{z}$: then

$$n^{2}z^{n-1} - A(n-1)^{2}z^{n-2} + B(n-2)^{2}z^{n-3} - C(n-3)^{2}z^{n-4} =$$

$$n(z-1)^{n-1} + n.(n-1).z(z-1)^{n-2} \ldots\ldots\ldots\ldots (2)$$

which when $z = 1$ or $z - 1 = 0$, gives

$$n^{2} - A(n-1)^{2} + B(n-2)^{2} - C(n-3)^{2} + \&c. = 0.$$

Multiply the expreſſion (2) by z, take fluxions and divide by $\dot{z}$: then

$$n^{3}z^{n-1} - A(n-1)^{3}z^{n-2} + B(n-2)^{3}z^{n-3} \&c. = n.(z-1)^{n-1}$$

$$+ 3n(n-1)(n-2).z(z-1)^{n-2} + n(n-1)(n-2).z^{2}(z-1)^{n-3} (3)$$

which when $z = 1$ or $z - 1 = 0$, gives

$$n^{3} - A(n-1^{3}) + B(n-2)^{3} - C(n-3)^{3} + \&c. = 0.$$

In like manner by continually repeating the ſame operations we ſhall at length arrive at a value of the expreſſion

$$n^{m}z^{n-1} - A(n-1)^{m}z^{n-2} + B(n-2)^{m}z^{n-3} - \&c.$$

which value it is manifeſt will involve $(z-1)$ in each of its terms when m is leſs than n: therefore, putting $z = 1$, or $z - 1 = 0$, we ſhall have, when m is leſs than n

$$n^{m} - A(n-1)^{m} + B(n-2)^{m} - C(n-3)^{m} + \&c. = 0.$$

This method has the advantage of being applicable to the valuation of the expreſſion $n^{m} - A(n-1)^{m} + B(n-2)^{m} - \&c.$ whether m is leſs than n or not.

Cor. $1 - n + n.\frac{n-1}{2} - n.\frac{n-1}{2}.\frac{n-2}{3} + \&c.$ till 1, = 0.

Third Solution, *by Mr.* Lowry.

In the ſerie n^{m}, $(n-1)^{m}$, $(n-2)^{m}$, &c. where the quantities n, $n-1$, $n-2$, &c. are in arithmetical progreſſion, the mth *order of differences* are equal (ſee Emerſon's Differential Method, Prop. 4); and, therefore, all the orders of differences higher than the *m*th muſt be equal to o: but by the queſtion *n* is greater than *m*, therefore, the *n*th order of differences in the above ſeries muſt be equal to o. Now (by Prop. 1. ibid) the given ſeries

$$n^m - n \cdot (n-1)^m + n\frac{n-1}{2} \cdot (n-2)^m - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}(n-3)^m + \&c. \text{ is}$$

the firſt term of the *n*th order of differences in the ſeries $n^m + (n-1)^m + (n-2)^m$, &c. and which is ſhewn above to be = to o, therefore,

$$0 = n^m - n \cdot (n-1)^m + n \cdot \frac{n-1}{2} \cdot (n-2)^m - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}(n-3)^m + \&c.$$

FOURTH SOLUTION, *by* PORUS.

By the binomial $(1-1)^n$ is $= 1 - n + n \cdot \frac{n-1}{2} - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} + \&c.$ and by Simpſon's celebrated theorem (Mathematical Diſſertations, page 62) if the terms of this ſeries be reſpectively multiplied by any quantities $p, q, r, s,$ &c. and the differences of theſe quantities be continually taken, and the firſt difference of the firſt order be denoted by D_1, the firſt of the ſecond order by D_2, the firſt of the third order by D_3, and in general the firſt of the *n*th order by D_n, the ſeries

$$p - nq + n \cdot \frac{n-1}{2} \cdot r - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot s + \&c. \text{ thence ariſing will be} =$$

$$p(1-1)^n + nD_1 \cdot (1-1)^{n-1} + n \cdot \frac{n-1}{2} D_2 \cdot (1-1)^{n-2}$$

$$+ n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} D_3 \cdot (1-1)^{n-3} + \&c.$$

Now it is manifeſt that all the terms of this latter ſeries vaniſh except the one, in which the index of $(1-1)$ is o, for $(1-1)^z$ is equal to nothing in every caſe but one, *viz.* when $z = 0$, and then $(1-1) = 0^0 = 1$; conſequently the ſeries

$$p - nq, \&c. \text{ is} = n \cdot \frac{n-1}{2} \ldots \frac{n-(n-1)}{n} \times D_n = D_n,$$

from which general theorem the propoſed one is directly deducible. For if $p = n^m$, $q = (n-1)^m$, $r = (n-2)^m$, $s = (n-3)^m$ &c. it is well known, and might be eaſily demonſtrated by means of the binomial theorem, that all their differences of the *m*th order are equal to each other, and, conſequently, all their differences of a higher order vaniſh; but m is leſs than n by the hypotheſis, therefore, D_n is $= 0$, and becauſe $D_n = p - nq + \&c$, we have by ſubſtitution

$$0 = n^m - n.(n-1)^m + n.\frac{n-1}{2}(n-2)^m - n.\frac{n-1}{2}.\frac{n-2}{3}$$

$(n-3)^m + \&c.$ ....*Q. E. D.*

SCHOLIUM. From the general theorem; $p - nq + n.\frac{n-1}{2}.r - \&c. = D_n$, a number of very curious properties may be derived; but the ſubject is foreign to the preſent queſtion.

The Medal for ſolving the Prize Queſtion is decided in favour of Mr. BAZLEY, *who will pleaſe to ſend for it to Mr.* GLENDINNING'S.

NOTICES

RELATING TO MATHEMATICS.

I. *Prize Queſtion of the National Inſtitute of France.*

" To give a Theory of the Perturbations of the Planet Pallas, diſcovered by Dr. OLBERS."

The labours of Mathematicians have now ſo far ſucceeded in overcoming the difficulties that occurred in the celebrated Problem of the Three Bodies, as to be able to determine with the requiſite exactneſs, the irregularities of the Moon, and of the old Planets, ariſing from their mutual Perturbations. In the caſe of all the old Planets, as well as in that of the Georgium Sidus diſcovered about 20 years ago by Dr. Herſchell, the excentricities of the orbits are ſmall, and the inclinations to the plane of the ecliptic are confined within narrow limits; circumſtances very favourable to abridge the computation of their numerous inequalities. The Planet Pallas, lately diſcovered by Dr. OLBERS, of which the excentricity is greater than that of Mercury, the moſt excentric of all the old Planets, and the inclination no leſs than 35° 37′, or more than five times greater than that of any other Planet, offers a new caſe of the Problem of the Three Bodies to the conſideration of Aſtronomers, for which the formulæ inveſtigated for the old Planets can be of little uſe. The intention of the Mathematical Claſs of the Inſtitute in propoſing the preſent ſubject for its Prize, is to call the attention of Mathematicians and Aſtronomers to this new caſe of the Problem, and to invite them to develope and diſcuſs the formulæ, in order to give them the degree of preciſion neceſſary in the actual circumſtances of the new Planet. As the Planet has not been obſerved for a ſufficient length of time to determine with accuracy the elements of the orbit, it is required that the competitors for the Prize, ſhall give the analytical coefficients independent of numerical values. Thus every new diſcovery in nature requires a correſpondent improvement

ment in theory. It is not difficult to foreſee that Mathematicians will be obliged to find out methods of reſolving the Problem of Perturbations entirely new; and that the wants of Aſtronomy will not be completely ſatisfied until a ſolution be accompliſhed that ſhall be independent of the elements of the orbit, and that ſhall converge only by the powers of the disturbing force.

II. NEW PLANET.

On the firſt of September laſt Mr. Harding, at the Obſervatory at Lilienthal, near Bremen, diſcovered a moveable star, in the ſign Pisces, in light and apparent magnitude ſimilar to the new Planet Ceres, and moving weſtward with a retrograde motion and increaſing ſouth declination. This Star is now conſidered to be a new Planet, of the Claſs of Ceres and Pallas, and in the ſame region. It has the appearance of a ſtar of the eighth magnitude. Its place, as obſerved by Dr. Olbers, on Sept. 8th. $8^h\ 11^m\ 20^s$ M. T. was

Right aſc.	Declin. South
$1^\circ\ 29'\ 39''$ ……………………	$0^\circ\ 47'\ 19''$.

the daily motion in A. R. $7'\ 16''$ or $31{\cdot}7''$ in time retrograde, the daily motion in Declin. $12'\ 34''$ South.

In the Philoſophical Magazine for October a chart of the apparent path of this new Planet is given.

III. DEATH OF MECHAIN.

On September the 20th, 1804, died the celebrated French Aſtronomer M. Mechain. He was employed with M. Delambre in the meaſurement of the Meridian from Dunkirk to Barcelona, which they completed in 1798. In 1803 he ſet out to continue the Meridional meaſurement as far as the Balearian Iſles; and he has fallen a victim to the labour of his undertaking, and to the unhealthineſs of the coaſt of Valentia in Spain.

IV. MATHEMATICAL WORKS LATELY PUBLISHED.

The Philoſophical Tranſactions of the Royal Society of London. Abridged by Charles Hutton, L.L.D. F.R.S. George Shaw, *M.D. F.R.S. F.L.S.* Richard Pearſon, M.D. F.A.S.

The original work, of which the preſent is an abridgement, extends to about 100 volumes, and contains a maſs of information not to be equalled by the proceedings of any other Society in the world. A work thus made up of the labours of the moſt learned men in Britain, as well as many in other parts of Europe, for a century and a half paſt, muſt be of infinite importance to all who are deſirous of becoming acquainted with the origin and preſent ſtate of human knowledge. Nothing therefore could be more auſpicious to the cauſe of science than the preſent undertaking, which propoſes to bring within a reaſonable compaſs all that is truly valuable in the original work. Four volumes of the abridgement have now appeared, and have fully juſtified the public expectation, a circumſtance which will not appear remarkable when it is recollected that the execution of the work is confided to men of the higheſt eminence in the departments of Science they have engaged to ſuperintend.

An Account of the Aſtronomical Diſcoveries of Kepler; including an Hiſtorical Review of the Syſtems which had ſucceſſively prevailed before his Time. By Robert Small, D.D. F.R.S.E.

The Principles of Analytical Calculation. By Robert Woodhouſe, A.M.

An Analyſis of the Principles of Natural Philoſophy. By Mathew Young, D.D. S.F. T.C.D. *late Biſhop of Clonfort.*

A Supplement to a Treatiſe on the Conſtruction and Properties of Arches, publiſhed in 1801. *By* George Atwood, Eſq.

The Philoſophical Tranſactions of the Royal Society of London for 1804. *Part I.*

A Collection of Mathematical Tables, for the uſe of Students; for the Practical Navigator, Geographer, and Surveyor; for Men of Buſineſs, &c. By Andrew Mackay, L.L.D. &c.

The Elements of Natural and Experimental Philoſophy. By Tiberius Cavallo, F.R.S.

A Practical Treatiſe on Perſpective, on the Principles of Dr. Brook Taylor. By Edward Edwards.

Elements of Science and Art: being a familiar Introduction to Natural Philoſophy and Chemiſtry, together with their Application to a Variety of elegant and uſeful Arts. By John Imiſon.

Meteorological Obſervations and Eſſays. By John Dalton, *Profeſſor of Mathematics and Natural Philoſophy, at the New College, Mancheſter.*

The Complete Navigator; or, An eaſy and familiar Guide to the Theory and Practice of Navigation; with all the requiſite Tables: illuſtrated with Engravings. By Andrew Mackay, L.L.D. F.R.S.

Elements of Natural Philoſophy; explaining the Laws and Principle of Attraction, Gravitation, Mechanics, Pneumatics, Hydroſtatics, Hydraulics, Electricity, and Optics: with a general View of the ſolar Syſtem, adapted to public and private Inſtruction: illuſtrated with Engravings. By John Webſter.

The Elements and Practice of Menſuration and Land Surveying; with an Appendix, containing Rules for meaſuring Hay-Stacks, Marl-pits, and Canals; with numerous Figures and Copper-plates. By Joſeph Becket.

The Tranſactions of the Royal Iriſh Academy. Vol. IX.

Tranſactions of the American Philoſophical Society. Vol. IV. and V.

V. MATHEMATICAL WORKS IN THE PRESS.

The fifth volume of the Tranſactions of the Royal Society of Edinburgh will ſpeedily be publiſhed.

Nearly ready for the preſs, and will be publiſhed early the enſuing year.

A Mathematical and Philoſophical Gloſſary: *exhibiting the derivations of words in the various branches of Mathematics and Philoſophy from their original languages: with their accentuation, and accurate pronunciation according to grammatical propriety, and uſage of the moſt eſteemed Authors. The whole being ſo arranged, that the words appropriate to each ſubject may be found collected alphabetically under the general term, each conciſely explained, and a reference for farther elucidation to the moſt explicit writers.* By Henry Clarke, L.L.D. *Profeſſor of Experimental Philoſophy in the Royal Military College, Marlow.*

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ARTICLE III.

Solutions to Queſtions propoſed in Number II.

I. QUESTION 61, *by Mr.* Thomas Squire, *Baldock.*

A tree growing upon the ſide of a hill which riſes due North at an angle of 30°, had the upper part blown off 12 feet from the ground by a ſtrong W. S. W. wind. It is required to find the height of the tree ſuppoſing it to ſtand perpendicular to the horizon, and the top (before the other part left the tree) to ſtrike the ground 40 feet from its bottom ?

First Solution, *by Mr.* Thomas Boole.

Let AEC (fig. 106, pl. 6.) repreſent the tree broken at E, AB an horizontal line in the direction of the foot of the hill, and C the point where the top of the tree falls upon the ſide of the hill. Conceive the line CD to be drawn perpendicular to the horizontal plane ABD, join AD, and draw DB perpendicular to AB; join alſo B, C, and A, C. Then there are three right angled triangles ADB, BDC, and ADC: the firſt being on the plane of the horizon, the ſecond on the plane of the meridian, and the third on a plane in the direction of the wind. And there are given $\angle \mathrm{BAD} = 22^\circ\ 30'$ $\angle \mathrm{DBC} = 30^\circ$ $\mathrm{AE} = 12$ and $\mathrm{AC} = 40$, to find EC; and this will be eaſieſt done by firſt finding the $\angle$ DAC. Let AK be taken = to the radius in the tables, and draw KI, KL parallel to DB, DC reſpectively, and join IL, which will evidently be parallel to BC; then KI will repreſent the ſine of $22^\circ\ 30'$ and rad. : tang. 30° :: KI : KL $=$ ſine $22^\circ\ 30' \times$ tang. 30°. But LK is the tang. of $\angle$ LAK to the ſame radius AK; therefore, the log. tang. of $\angle$ LAK $=$ (DAC $=$) the log. ſine of $22^\circ\ 30'$ $+$ log. tang. 30° $=$ tang. $12^\circ\ 27'\ 30''$ nearly. Hence $\angle \mathrm{EAC} = 90^\circ - 12^\circ\ 27'\ 30'' = 77^\circ\ 32'\ 30''$ and the two ſides AC and AE being given, EC is found $= 39{\cdot}204$, and conſequently $12 + 39{\cdot}204 = 51{\cdot}204$ feet, nearly the height of the tree required.

SECOND SOLUTION, *by Mr.* JOHN SMITH, *Alton Park.*

Let ABC (fig. 107, pl. 6.) be a horizontal ſection, ABD a vertical ſection in a north direction, and ACE a vertical ſection in E. N. E. direction, or contrary to that of the wind, all paſſing from the foot A of the tree AFG through the hill. In the right angled triangles ABC, ABD are given the angle $BAC = 67° 30'$ and the angle $BAD = 30°$, whence, by trigonometry, AB, AC, and BD ($= CE$) are reſpectively as 1, 2·613126 and ·5773503; and in the right angled triangle ACE the ratio of the ſides AC to CE is given, viz. as 2·613126 to ·5773503, therefore the angle CAE is $= 12° 27' 32''$, the angle of the hill's declivity in the direction in which the upper part of the tree fell. Now in the triangle AFG, AF, AG, and the angle GAF are given, viz. $AF = 12$, $AG = 40$, and the angle GAF ($= \angle CAF - \angle CAE = 90° - 12° 27' 32''$) $= 77° 32' 28''$; therefore FG is eaſily found $= 39·2032$ the length of the piece broken off. Hence $AF + FG = 51·2032$ feet, the whole length of the tree.

Solutions were alſo received from Meſſrs. Butterworth, Cavill, Collins, Dawes, Jones, Myers, N. R. D., Pickering, Ryley, *and* Squire.

II. QUESTION 62, *by Mr.* JOHN CAVILL, *Beighton.*

In any plane triangle it will be as the baſe is to the difference between the ſum of the ſides and baſe, ſo is the rectangle under half the perimeter and that part of the diameter of the circumſcribing circle which is perpendicular to the baſe, and falls below it, to the area of the triangle. Required a demonſtration ?

SOLUTION, *by Mr.* THOMAS MYERS.

Let ACB (fig. 108, pl. 6.) be any plane triangle, FED the diameter of the circumſcribing circle perpendicular to the baſe AB, and biſecting it in E. Join DC, DA, and let O be the centre, and OP the radius of the inſcribed circle. Then the triangles DEA, OPC are ſimilar, and therefore AE : CP :: ED : OP, or AE : CP :: ED × half the perimeter : OP × half the perimeter (or the area of the triangle ACB). But it is well known that CP is equal to half the difference between the ſum of the ſides and the baſe, and, therefore, doubling the two firſt terms of the above proportion, AB is to the difference between the ſum of the ſides and the baſe, as ED × half the perimeter to the area of the triangle. *Q. E. D.*

Thus the propoſition was demonſtrated by Meſſrs. Bazley, Boole, Butterworth, Cavill, Collins, Dawes, Johnson, Jones, Mellor, Pickering, Roville *and* Ryley.

III. QUESTION 63, *by* HYPATIA.

A circular ring revolves on the inſide of another exactly double its ſize; required the locus of a given point in its circumference?

FIRST SOLUTION, *by Mr.* S. JONES, *Irondiſh.*

In fig. 109, pl. 6. let P be the place where the given point in the leſſer ring is in contact with the greater ring, and let C be the point of contact of the rings when the given point is in ſome other poſition as at p, and join OC, Op, and Pp. Then ſince every point of the arc pC is ſucceſſively applied to the arc PC, it is evident that the arcs pC, PC are equal, and becauſe the diameter of the circle APC is double the diameter of the circle OpC, it follows that the angle COp on the arc pC is equal to the ∠COP at the centre on the arc PC; therefore the points O, p, P muſt be in the ſame ſtraight line, and conſequently the locus of p is the diameter AP.

COR. If pQ be a diameter of the leſſer ring, the locus of Q will be the diameter ab, drawn perpendicular to the diameter AP.

SECOND SOLUTION, *by Mr.* G. WILLIAMS.

Let ACB (fig. 110, pl. 6.) repreſent the greater ring, and AO the leſs. Let AO revolve to the poſition OA′C, and the point A will move from A to A′ along the diameter AOB. For ſuppoſe A, by the revolution of the ring AO, moves to any point a

without the diameter AB, and draw the radius G*a* meeting AB in *n*. Then ſince the circumference AC is equal to the circumference *a*C and the radius OA (by the queſtion) equal to twice the radius GC, the angle CG*a* is equal to twice the angle COA, conſequently, G*n* = GO = G*a*, which is abſurd. Hence if the ring AO revolve along the ſemidiameter ACB, the point A will be always in AB, which is the locus required.

According to one or other of theſe Methods were the anſwers given by Meſſrs. Bazley, Boole, Butterworth, Cavill, Dawes, Johnſon, N. R. D., Roville *and* W. Smith.

IV. QUESTION 64, *by Mr.* THOMAS HOPPER, *Mancheſter.*

Admitting the latitude of Mancheſter to be 53° 25′ N. It is required to find the ſituation of two other places on the ſurface of the globe, ſuch, that when it is noon at Mancheſter on the third day of March, 1805, the ſun's altitude may be equal at all the three places; and the firſt place from Mancheſter may bear Eaſt-South-Eaſt, and the ſecond place from the firſt Weſt-North-Weſt?

FIRST SOLUTION, *by Mr.* JOHN DAWES.

Fig. 111. pl. 6. Let the primitive circle repreſent the horizon; the right circle MN the meridian of Mancheſter; P the pole; Z the zenith; ⊙ the ſun on the Meridian; R and S the firſt and ſecond places required; let Z⊙ = R⊙ = S⊙ = the given zenith diſtance of the ſun, which on account of the difference of longitude of Mancheſter and Greenwich is = 60° 16′ 46″; let the ∠ ⊙ZR be the given bearing of the firſt place from Mancheſter, and the ∠ PRS that of the firſt and ſecond place, each = 67° 30′; now the △ ⊙ZR (being iſoſceles) is known, and becauſe ZR is known, therefore the △ PZR is known: but becauſe each of the two places bear under the ſame angle, *i.e.* ∠ ⊙ZR = PRS = ⊙RZ, therefore PRZ = ⊙RS = 33° 34′ 42″, wherefore the iſoſceles △ R⊙S is given; and becauſe RS is given = 111° 9′ 43″, therefore the △ PRS is given: from theſe data,

the latitude of R the firſt place (comp. RP) = 5° 23′ 51″
the diff. of long. of Mancheſter and R = ZPR = 59 8 28
the latitude of S the ſecond place (comp. PS) = 18 45 00
the diff. of long. of Mancheſter and S = ZPS = 55 22 38.

Second Solution, *by* N. R. D.

Let M (fig. 112, pl. 6.) be the place on the ſphere anſwering to Mancheſter, Z and Q the other two places where the ſun's meridian altitude is the ſame as at Mancheſter, the place Z bearing E.S.E. from M, and the place Q bearing W. N. W. from Z.

Let P be the pole and S the ſun in the meridian PM, and deſcribe the arcs MZ, PZ, SZ, QZ, and QP. Then, becauſe of the equal altitudes, the arcs SM, SZ, and SQ are equal; and the ſun's declination on the third of March, 1805, being 6° 51′ 55″ ſouth, this added to the co-latitude of Mancheſter gives SM = SZ = SQ = 60° 16′ 55″. Again, as Z is E. S. E. of M, the angle SMZ is 67° 30′, and in the iſoſceles Δ SMZ the ∠ MSZ is found = 79° 45′ 41·4″. Alſo in the Δ PSZ we have PS = 96° 51′ 55″, SZ = 60° 16′ 55″, and ∠ MSZ = 79° 45′ 41·4″, to find PZ = 84° 36′ 21″, and the angle SPZ = 59° 8′ 33·8″. Hence the place Z is in North latitude 5° 23′ 39″, and in longitude 59° 8′ 33·8″ E. from Mancheſter. Again, becauſe Q is W. N. W. from Z, the ∠ PZQ = SMZ = SZM = 67° 30′, hence ∠ PZS = 101° 4′ 29″, and ∠ SZQ = SQZ = 33° 31′ 29″; and in the iſoſceles Δ ZSQ, the ſide QZ is found = 111° 9″ 39″. Alſo in the Δ QZP the ſide QZ, PZ, and the included ∠ PZQ are given to find QP = 71° 15′ 5″ and ∠ ZPQ = 115° 31′ 13″: hence ∠ QPS = 55° 22′ 40″, the longitude of Q weſt from Mancheſter, and the latitude 18° 44′ 55″ North.

Meſſrs. Boole, Collins, *and* Hopper, *the Propoſer, alſo anſwered this Queſtion.*

V. QUESTION 65, *by* Arithmeticus.

Theorem. Let N be any number whatever, and let $a =$ the difference of N, and the next greater ſquare number, and $b =$ the difference of N, and the next leſs ſquare number. Then $N - a \times b$ will be always a ſquare number?

Solution, *by Mr.* Samuel Ryley.

Let $N + a = (x + 1)^2$, then, by the queſtion,
$N - b = x^2$;
therefore $N = (x + 1)^2 - a = x^2 + b$,
and $a = 2x + 1 - b$,
or $-ab = -2bx - b + b^2$;

and, therefore, $N - ab = x^2 + b - 2bx - b + b^2 = (x - b)^2$, which is manifeſtly a ſquare number; or, if for x there be put its value, found from the equation $a = 2x + 1 - b$, we have $x - b = \frac{a - b - 1}{2}$, and, therefore, $N - ab = \left(\frac{a - b - 1}{2}\right)^2$ a ſquare number, as before.

Cor. If $N = ab$, x will be $= b$, and in this caſe $a = b + 1$ and $N = b^2 + b$; hence if N be compoſed of a number added to its ſquare, it follows that $N - ab$ will always be $= 0$.

Thus if $N = 2^2 + 2 = 6$, then $b = 2$ and $a = 3$; and
$N - ab = 6 - 3 \times 2 = 0$.

Alſo, if $N = 3^2 + 3 = 12$, then $b = 3$ and $a = 4$; and
$N - ab = 12 - 3 \times 4 = 0$.

And again, if $N = 4^2 + 4 = 30$, then $b = 4$ and $a = 5$; and $N - ab = 20 - 5 \times 4 = 0$, &c. &c.

And thus it was anſwered by Meſſrs. Bazley, Boole, Butterworth, Cavill, Collins, Elliott, Limenus, Myers, N. R. D., Pickering, *and* W. Smith.

VI. QUESTION 66, *by Mr.* S. WADE.

At what time in latitude 50° 31′ (the ſun being in the equinox) will the ſhadow of the ſtile on the plane of a direct weſt vertical ſun-dial be a minimum?

SOLUTION, *by Mr.* JOHN DAWES, *Birmingham.*

When the ſhadow is a minimum it will fall on the ſix o'clock line upon the plane, which on account of refraction will not happen at the time the ſun ſets, nor when it apparently ſets, but ſome where between thoſe times; let EB (fig. 113, pl. 6.) be the true, and SC the apparent path of the ſun; the refraction for this ſmall interval of time may be ſuppoſed conſtant, therefore, when the ſun is in the horizon at ⊙, it will appear at S, and when S moves to C the ſhadow is the minimum required; therefore in the quadrantal △ ⊙SP are given ⊙S = 33′ min. and ∠ P⊙S = 39° 29′ to find the ∠ ⊙PS = 20′,, 59″, which gives 1m 23s 56th. after ſix the time required.

Meſſrs. Boole, Collins, Elliott, *and* Pickering, *ſent Solutions to this Queſtion.*

VII.

VII. QUESTION 67, *by Mr.* BENJAMIN BEVAN.

In a plane triangle, let w be the centre of a circle paſſing through x, y, and z; then will $Cw = Cc$, and be in the ſame right line; and $wx = wy = wz = 2R$, or the diameter of the circumſcribing circle; where x, y, z, &c. repreſent the ſame points and lines as they denote in the Synopſis of Data, for the Conſtruction of Triangles?

SOLUTION, *by Mr.* JOHN BUTTERWORTH, *Haggate.*

Let S, H and G (fig. 114, pl. 6.) be the points where the lines xc, yc, and zc meet the circumſcribing circle, and draw the radii CS, CH, and CG; alſo draw xw parallel to CS meeting cC produced in w, and join yw, zw; then w is the centre of a circle paſſing through x, y and z. For it is now well known that $Sc = Sx$, therefore $Cc = Cw$ and $CH = Hy$, conſequently yw is parallel to CH. But CH is $=$ CS, therefore yw is $= wx = 2CS$. In like manner it is proved that $zw = xw = 2CS$; therefore w is the centre of a circle paſſing through the points x, y and z, and conſequently the points c, C, w, are in a ſtraight line, and $Cc = Cw$; and $yw = xw = zw = 2CS = 2R$. *Q. E. D.*

Thus nearly was the propoſition demonſtrated by Meſſrs. Boole, Dawes, *and* Johnſon.

VIII. QUESTION 68, *by* ALCOR.

Let A and B be two given points, CD, CE, two ſtraight lines given by poſition; let AF, BF be inflected to F any point in one of the given lines, and let BF meet CD in G, draw GH parallel to AB meeting AF in H, then H will be in a ſtraight line given by poſition. Required the demonſtration?

FIRST SOLUTION, *by Mr.* JOHN BUTTERWORTH.

Fig. 115, pl. 6. Draw BL parallel to CD, meeting CE in L, and join AL, CH; then ſince HG is parallel to AB and BL parallel to CD, we have AB : HG :: BF : FG :: BL : CG, and by permutation AB : BL :: HG : CG; therefore the triangles ABL, HGC are ſimilar, and, therefore, CH is parallel to AL: *But AB and BL* are given lines; therefore, CG has to CH

GH a given ratio, and HG makes a given angle with CG which is given by poſition; wherefore the locus of the point H is the ſtraight line CH, and which is alſo given by poſition, becauſe it paſſes through a given point C, and is parallel to AL, which is given by poſition.

SECOND SOLUTION, *by Mr.* W. SIMPSON, *Bolton.*

Fig. 115, pl. 6. Draw BL parallel to CD meeting CE in L, and join AL; draw CH parallel to AL, and it will be the locus ſought.

Becauſe BL is drawn from a given point parallel to a line CD, given in poſition and meeting CE, alſo given in poſition, in L, the point L will be given, and conſequently AL will be given by poſition, and therefore CH will alſo be given by poſition; we have therefore only to prove that HG is parallel to AB. Now, by parallels, AF : HF :: LF : CF :: BF : GF, or AF : HF :: BF : GF, therefore HG is parallel to AB.

COR. If D*h* be drawn parallel to AB meeting BG produced in *h*, the locus of the point *h* is alſo a ſtraight line given by poſition.

Ingenious ſolutions were alſo received from Meſſrs. Bazley, Boole, Collins, Mellor, Myers, Pickering, Roville *and* W. Smith.

IX. QUESTION 69, *by* GEOMETRICUS.

Four ſtraight lines are given by poſition, it is required to deſcribe a triangle whoſe angles may be upon three of the lines, and whoſe ſides may make given angles with the remaining line?

SOLUTION, *by* N. P. O. ROVILLE.

Let AB, CD, AE (fig. 116, pl. 6.) be the three ſtraight lines given by poſition, on which the angular points of the triangle are to be, and AH the other line with which the ſides of the triangle are to make given angles. Draw any line EG to make one of the given angles with AH, and from the points G and E where it meets AB, AE, draw GF and EF to make with AH the other two angles; draw alſo AF to interſect CD in Q: Now let QP, QR be drawn parallel to EF, and FG reſpectively, and join PR, then PQR is the triangle required.

Becauſe QP is parallel to EF, and QR to FG, it is evident that PQ and QR make the given angles with AH; and AE : PE :: AF : QF :: AG : RG; therefore PQ is parallel to EG; and, conſequently, PR makes alſo the given angle with AH.

Meſſrs. Bazley, Boole, Butterworth, Collins, *and* Simpſon, *likewiſe sent neat anſwers to this Queſtion.*

X. QUESTION 70, *by the Rev.* L. EVANS, *Royal Academy, Woolwich.*

It is required to biſect an arch of a circle by the deſcription of circles only?

SOLUTION, *by the Proposer.*

Let BC (fig. 51, pl. 2.) be the given arc, and A the centre; with the centres B, C, and diſtances BA or CA, the radius of the circle deſcribe two arcs AD, AE, each equal to the given arc BC. With the centres D, E, and diſtance DC = BE deſcribe two arcs to interſect at F; then with the radius AF, and the ſame centres D, E, deſcribe two other arcs which will interſect in the circumference at G, and the arc BC will be biſected at the point G.

DEMONSTRATION. The ſides of the three triangles DBA, BAC, ACE, being reſpectively equal, we have the angle BCA equal to the angle CAE; therefore BC is parallel to AE, and therefore BAEC is a parallelogram. It is proved in the ſame manner that BCAD is a parallelogram, and the diagonal AB is equal to the oppoſite ſides BD, AC; therefore DC^2 (DF^2) $= AB^2 + 2AD^2$. Again, becauſe DA, AE, are parallel to BC, the points D, A, E, are in the ſame right line, and the triangles FAD, FAE having all their ſides equal, have alſo the angles FAD, FAE equal, conſequently they are right angles. In the ſame manner it is ſhewn that the angles DAG, EAG, are right angles. Therefore, $DF^2 = DA^2 + AF^2 = DA^2 + DG^2$, and $DG^2 = AD^2 + AG^2$; wherefore $DF^2 = AG^2 + 2AD^2$; conſequently, $AB^2 = AG^2$ and $AB = AG$ the radius of the circle; therefore, G is in the circumference of the circle. And becauſe AG is perpendicular to DE it will alſo be perpendicular to BC, which is parallel to DE, and therefore the arc BC is biſected in G.

Meſſrs. Boole, Cavill, Collins, Myers, *and* Pickering, *anſwered this Queſtion.*

XI. QUESTION 71, *by* WHACUM.

Let AB be a given ſtraight line, and P a given point without it, let CPD be drawn meeting AB in C, and let CP be to PD as AC to CB. Required the locus of the point D?

FIRST SOLUTION, *by Mr.* BAZLEY, *Bolton.*

Join AP (fig. 117, pl. 6.) and produce BA till AS = AB, draw SF parallel to AP, and PF parallel to AS cutting SF in F; alſo draw DOE parallel to AP meeting PF in O and SA in E; and laſtly draw BG parallel to AP meeting PC produced in G. Then, by the ſimilar triangles ACP and BCG, AC : CB :: CP : CG, and by hypotheſis AC : CB :: CP : DP, therefore CG is equal to DP; and the triangles BCG, OPD being equal as well as ſimilar; therefore CB = OP = AE, and ſince AS = SB, SE will be equal to AC. Again, ſince the triangles CPA, CDE are ſimilar we have CE : CA :: ED : AP, therefore EC × AP = CA × ED, that is SA × AP = SE × ED; conſequently the locus of D is an hyperbola deſcribed through the given point P to the aſymptotes SF, SA.

If, (as in fig. 118, pl. 6,) the point D be taken on the other ſide of C, produce AP to Q, ſo that PQ may be equal to AP, and complete the rectangles AQSB, and APFB, and let SB produced meet PC produced in G, and draw DH parallel to AB and DO parallel to SB; then, as above, CG = PD, CB = OP, and DH = AC; alſo GH = AP = BF = SF, GF = GH + HF = AP + HF = PQ + HF = SH. Hence by the ſimilar triangles GDH, GPF; GH : HD :: GF : FP; but GH = AP = PQ = SF, and GF = SH; therefore SF : HD :: SH : FP; conſequently, SF × FP = SH × HD; and therefore the locus of the point D is an hyperbola deſcribed through the point P to the aſymptotes QS, SB, which is the very ſame hyperbola as before, only differently ſituated.

COR. BP is a tangent at P. For when CP falls in with BP there is no interſection of the curve made by BP, and, conſequently, BP touches it in P. This agrees with what is commonly ſhewn by writers on conics; for in fig. 117, BA = AS, and in fig. 118, BF = FS, and therefore by conic ſections BP is a tangent at P.

SECOND SOLUTION, *by Mr.* W. SMITH.

Produce AB till AS = AB, (fig. 117, pl. 6.) and draw SI parallel to AP; then a hyperbola deſcribed through the point P to the aſymptotes SB, SH will be the locus required when the point D is above the point P, but if it is below the point P (as in fig. 118, pl. 6,) produce AP till PQ is = to AP, and draw QS parallel to AR meeting BS drawn parallel to AP in S; then a hyperbola deſcribed through the point P to the aſymptotes SQ, SR will, in this caſe, be the locus required.

DEMONSTRATION. Draw the line PCD to any point in the curve, and let PF, DH, be drawn parallel to AB, the latter meeting AP produced in N; then, by the property of the hyperbola

HD × HS = AP × FP = AP × HN, or

HS = AP + PN : AP :: HN : HD, and by diviſion

PN : AP :: DN : HD; but by ſimilar triangles

PN : AP :: DN : AC: therefore HD = AC;

and ſince HN = AB, DN is = to BC; and, conſequently, AC : CB (DN) :: CP : PD. It is therefore manifeſt that the hyperbola PD is the locus required.

Ingenious Solutions were alſo received from Meſſrs. Boole, Elliott, Limenus, *and* Simpſon.

XII. QUESTION 72, *by* BOMBELLI.

To divide $\frac{1}{4}$ into 5 parts ſuch, that either part added to the ube of $\frac{1}{4}$ ſhall be a ſquare?

FIRST SOLUTION, *by* LIMENUS, *Bruton, Somerſet.*

Let the five parts into which $\frac{1}{4}$ is required to be divided be deſignated by a, b, c, d, and e, and let the ſquare numbers to which thoſe parts when ſeverally increaſed by the cube of $\frac{1}{4}$ or $\frac{1}{64}$ are to be equal, be denoted by α^2, β^2, γ^2, δ^2 and ε^2, and we ſhall immediately have

$$a = \alpha^2 - \frac{1}{64},\ b = \beta^2 - \frac{1}{64},\ c = \gamma^2 - \frac{1}{64},$$

$$d = \delta^2 - \frac{1}{64},\ e = \varepsilon^2 - \frac{1}{64}.$$

Wherefore having determined α, β, γ, &c. the parts a, b, c, &c. will become known, and the only reſtriction dependent on thoſe equations is that α, β, γ, &c. be each greater than $\frac{1}{8}$, in order that all the parts may come out poſitive. The remaining condition is that the ſum of the parts be $\frac{1}{4}$, or that $\frac{1}{4} = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 - \frac{5}{64}$, or $\frac{21}{64} = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2$. It is therefore neceſſary to divide the fraction $\frac{21}{64}$ into five ſquares, and to effect ſuch a diviſion it will be requiſite to aſſume ſome five numbers that poſſeſs that property, and then we ſhall be enabled to obtain a general ſolution of the problem.

The denominator of $\frac{21}{64}$ is itſelf a ſquare, and the numerator 21 being but ſmall, it will readily be perceived that the only ways in which it is capable of being formed into five integral ſquares are by the numbers 16, 4, 1, 0, 0; 9, 9, 1, 1, 1; and 9, 4, 4, 4, 0. And each of theſe ſets will enable us to diſcover an infinite number of fractions poſſeſſing the required property; but that the ſame ſolution may be made applicable to them all, we will deſignate the roots of one of the ſets by m, n, p, q, r, ſo that $m^2 + n^2 + p^2 + q^2 + r^2 = \frac{21}{64}$.

Let v be ſome intermediate quantity, and l, k, λ, μ, ν, other component parts of the ſquares $\alpha^2, \beta^2, \gamma^2, \delta^2, \varepsilon^2$, ſuch that $\alpha = lv - m$, $\beta = kv - n$, $\gamma = \lambda v - p$, $\delta = \mu v - q$, and $\varepsilon = \nu v - r$. And by expanding the equation $\frac{21}{64} = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2$, and deducting from it that of $\frac{21}{64} = m^2 + n^2 + p^2 + q^2 + r^2$, there will reſult

$$-2v(ml + nk + p\lambda + q\mu + \nu r) + v^2(l^2 + k^2 + \lambda^2 + \mu^2 + \nu^2) = 0,$$

and thence $v = \frac{2ml + 2nk + 2p\lambda + 2q\mu + 2r\nu}{l^2 + k^2 + \lambda^2 + \mu^2 + \nu^2}$.

In this formula the quantities l, k, λ, μ, ν are aſſumable at pleaſure, and the quantities m, n, p, q, r, are to be determined by taking one of the ſets of numbers before given. Having thus found a value of v, the correſponding ones of α, β, γ, &c. and thence the required parts a, b, c, &c. will become known.

By making uſe of the firſt ſet, and taking $l = 1$, $k = 1$, $\lambda = -1$, $\mu = 3$, and $\nu = 2$, we ſhall find $v = \frac{5}{8}$, and the five parts in this caſe will be $\frac{665}{4096}$, $\frac{161}{4096}$, $\frac{105}{4096}$, $\frac{57}{4096}$ and $\frac{36}{4096}$. Alſo if we take $l = 3$, $k = 1$, $\lambda = -2$, $\mu = 3$, and $\nu = 2$, then $v = \frac{8}{9}$, and the five parts will come out $\frac{544}{5184}$, $\frac{495}{5184}$, $\frac{175}{5184}$, $\frac{63}{5184}$, and $\frac{19}{5184}$; and with the like facility may innumerable other ſolutions be derived.

SECOND SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

Let the parts be denoted by w, v, x, y, and z, and by the queſtion put

$$w+\frac{1}{64}=a^2,\ v+\frac{1}{64}=b^2,\ x+\frac{1}{64}=c^2,\ y+\frac{1}{64}=d^2 \text{ and } z+\frac{1}{64}=e^2.$$

By taking the ſum of all the equations,

$$w+v+x+y+z+\frac{5}{64}=\frac{1}{4}+\frac{5}{64}=a^2+b^2+c^2+d^2+e^2.$$

Now put

$$\frac{3}{8}-mu=a,\ \frac{3}{8}-nu=b,\ \frac{1}{8}+pu=c,\ \frac{1}{8}+qu=d, \text{ and } \frac{1}{8}+ru=e;$$

Then

$$\frac{21}{64}=a^2+b^2+c^2+d^2+e^2.$$

$$=\left(\frac{3}{8}-mu\right)^2+\left(\frac{3}{8}-nu\right)^2+\left(\frac{1}{8}+pu\right)^2+\left(\frac{1}{8}+qu\right)^2+\left(\frac{1}{8}+ru\right)^2$$

$$=\frac{21}{64}+u^2\times(m^2+n^2+p^2+q^2+r^2)+\frac{u}{4}\times\left\{p+q+r-3(m+n)\right\}.$$

From whence $u=\frac{3(m+n)-(p+q+r)}{4(m^2+n^2+p^2+q^2+r^2)}$, where m, n, p, q, and r, may be taken at pleaſure; but with this reſtriction, that each of the quantities a, b, c, d, and e, ſhall exceed $\frac{1}{8}$.

Example 1. Take $m=2$, $n=1$, $p=3$, $q=2$, and $r=1$; then $u=\frac{3}{4\times19}$; whence $a=\frac{3}{8}-mu=\frac{3}{8}-\frac{6}{4\times19}=\frac{45}{8\times19}$; $b=\frac{3}{8}-nu=\frac{3}{8}-\frac{3}{4\times19}=\frac{51}{8\times19}$; $c=\frac{1}{8}+pu=\frac{1}{8}+\frac{18}{8\times19}=\frac{37}{8\times19}$; $d=\frac{1}{8}+qu=\frac{1}{8}+\frac{12}{8\times19}=\frac{31}{8\times19}$; and $e=\frac{1}{8}+ru=\frac{1}{8}+\frac{6}{8\times19}=\frac{25}{8\times19}$. Hence again, $w=a^2-\frac{1}{64}=\frac{26}{361}$; $v=b^2-\frac{1}{64}=\frac{35}{361}$; $x=c^2-\frac{1}{64}=\frac{63}{4\times361}$; $y=d^2-\frac{1}{64}=\frac{75}{8\times361}$, and $z=e^2-\frac{1}{64}=\frac{33}{8\times361}$.

Example 2. Take $m=2$, $n=1$, $p=2$, $q=1$, and $r=4$; then $u=\frac{1}{4\times13}$; whence $a=\frac{3}{8}-mu=\frac{3}{8}-\frac{4}{8\times13}=\frac{35}{8\times13}$; $b=\frac{3}{8}-nu=\frac{3}{8}-\frac{2}{8\times13}=\frac{37}{13}$; $c=\frac{1}{8}+pu=\frac{1}{8}+\frac{4}{8\times13}=\frac{17}{8\times13}$; $d=\frac{1}{8}+qu=\frac{1}{8}+\frac{2}{8\times13}=\frac{15}{8\times13}$; and $e=\frac{1}{8}+ru=\frac{1}{8}+\frac{8}{8\times13}=\frac{21}{8\times13}$. And hence, $w=a^2-\frac{1}{64}=\frac{66}{4\times169}$; $v=b^2-\frac{1}{64}=\frac{75}{4\times169}$ $x=c^2-\frac{1}{64}=\frac{15}{8\times169}$; $y=d^2-\frac{1}{64}=\frac{7}{8\times169}$ and $z=e^2-\frac{1}{64}=\frac{17}{4\times169}$.

Again, in the equation $a^2+b^2+c^2+d^2+e^2=\frac{21}{64}$, put
$\frac{3}{8}-mu=a$; $\frac{2}{8}-nu=b$; $\frac{2}{8}-pu=c$; $\frac{2}{8}-qu=d$, and $ru=e$;
then $\frac{21}{64}=a^2+b^2+c^2+d^2+e^2$

$$=\left(\frac{3}{8}-mu\right)^2+\left(\frac{2}{8}-nu\right)^2+\left(\frac{2}{8}-pu\right)^2+\left(\frac{2}{8}-qu\right)^2+r^2u^2$$

$$=\frac{21}{64}-\frac{3mu}{4}-\frac{2nu}{4}-\frac{2pu}{4}-\frac{2qu}{4}+m^2u^2+n^2u^2+p^2u^2+q^2u^2+r^2u^2.$$

Whence $u = \frac{3m + 2 \times (n + p + q)}{4 \times (m^2 + n^2 + p^2 + q^2 + r^2)}$, where m, n, p, q and r may be taken at pleaſure, but ſubject to the ſame reſtriction as before.

For example, take $m = 2$, $n = 3$, $p = 2$, $q = 1$, and $r = 12$; then $u = \frac{1}{36}$; whence $a = \frac{3}{8} - mu = \frac{23}{4 \times 18}$; $b = \frac{2}{8} - nu = \frac{1}{6}$; $c = \frac{2}{8} - pu = \frac{7}{2 \times 18}$; $d = \frac{2}{8} - qu = \frac{2}{9}$; and $e = ru = \frac{1}{3}$. And hence $w = a^2 - \frac{1}{64} = \frac{7}{81}$; $v = b^2 - \frac{1}{64} = \frac{7}{16 \times 36}$; $x = c^2 - \frac{1}{64} = \frac{115}{64 \times 81}$; $y = d^2 - \frac{1}{64} = \frac{175}{64 \times 81}$, and $z = e^2 - \frac{1}{64} = \frac{55}{9 \times 64}$.

Solutions were alſo received from Meſſrs. Boole, Butterworth, Collins, Myers, Pickering, *and* Ryley.

XIII. QUESTION 73, *by* HYPATIA.

When the baſe of a triangle is given, and one of the angles at the baſe is double the other; what is the locus of the vertex?

FIRST SOLUTION, *by the Propoſer.*

Let ACB (fig. 119, pl. 6.) be the triangle having the baſe AB given, and the angle ABC double the angle BAC. Draw CD perpendicular to AB; take DF = DA and join CF; then the angle BAC will be equal to the angle BFC, therefore the angle ABC will be double BFC; but ABC is equal to the two angles BCF, BFC, therefore the angles BCF, BFC are equal, and hence the lines BC, BF are equal; now BF is equal to DF — DB, or to AD — DB, therefore BC is equal to AD — DB. Biſect AB in E and draw EG perpendicular to AB, and from C draw CH perpendicular to EG, then E is a given point, and EG a ſtraight line given by poſition; and becauſe AD — DB is equal to 2ED or to 2HC, therefore BC = 2HC or BC : HC :: 2 : 1, that is, BC has to HC a given ratio of a greater to leſs, hence it is evident that the locus of the point C is a hyperbola of which F is a focus and EG the directrix.

Cor. If upon AB any ſegment of a circle be deſcribed cutting the hyperbola which is the locus of the vertical angle in K, the part of the circumference AKB will be biſected in K. For join AK, BK, then the angle EBK being double the angle BAK, the circumference AK is double BK, therefore BK is one-third of BKA.

This queſtion and its application to the triſection of an arch conſtitute Prop. 34, Book iv. of the Math. Collect. of Pappus.

Second Solution, *by Mr.* W. Smith, *Liverpool.*

Biſect the baſe AB in E, and draw the indefinite perpendicular EG; divide EB in V, ſo that BV $=$ 2EV; with the focus B, vertex V, and directrix EG, deſcribe a hyperbola, and it will be the locus required.

Draw AC, BC to any point in the curve, and draw CH perpendicular to EG, and produce it till HI $=$ HA and join CI. Then by the property of the hyperbola EV : VB :: HC : CB, but VB $=$ 2EV, therefore CB $=$ 2CH $=$ IC: but ſince AE $=$ EB, IH $=$ HC, and EH perpendicular to AB and IC, it is obvious that AI is $=$ to BC and the angle BAI $=$ ABC; therefore IC is $=$ to AE and the angle ICA (BAC) $=$ IAC; and therefore the angle BAI or ABC is equal to twice the angle BAC.

This Queſtion was alſo anſwered by Meſſrs. Bazley, Boole, Collins, Elliott, Johnſon, Limenus, Myers, Pickering, *and* Simpſon.

XIV. QUESTION 74, *by Mr.* George Sanderson.

Determine the value of $\frac{\text{coſ.}\,\varphi}{\text{coſ.}\,\psi}$, when $\frac{\text{ſin.}\,\varphi}{\text{ſin.}\,\psi} = n$, a given number, and $\varphi + \psi = \alpha$, a given arch.

First Solution, *by* Scoticus.

Suppoſe $\frac{\text{coſ.}\,\varphi}{\text{coſ.}\,\psi} = x$, then $\text{coſ.}\,\varphi = x\,\text{coſ.}\,\psi$ and $\text{ſin.}\,\varphi = n\,\text{ſin.}\,\psi$, hence, putting P for $\text{coſ.}\,\varphi\,\text{coſ.}\,\psi$, and Q for $\text{ſin.}\,\varphi\,\text{ſin.}\,\psi$, we have $\text{coſ.}^2\,\varphi = xP$, $\text{ſin.}^2\,\varphi = nQ$, $nx\,\text{coſ.}^2\,\psi = nP$, $nx\,\text{ſin.}^2\,\psi = xQ$, therefore $xP + nQ = \text{coſ.}^2\,\varphi + \text{ſin.}^2\,\varphi = 1$, ...(1)
and $nP + xQ = nx\,(\text{coſ.}^2\,\psi + \text{ſin.}^2\,\psi) = nx$, ...(2)

The arithmetic of ſines furniſhes this other equation,

$$P - Q = \text{coſ.}\,(\varphi + \psi) = \text{coſ.}\,\alpha \quad \ldots\ldots\ldots\ldots (3)$$

From the firſt and third of theſe equations we find

$$(n + x)\,P = 1 + n\,\text{coſ.}\,\alpha,$$

and from the ſecond and third,

$$(n + x)\,P = x\,(n + \text{coſ.}\,\alpha).$$

Therefore, $x\,(n + \text{coſ.}\,\alpha) = 1 + n\,\text{coſ.}\,\alpha$,

$$\text{and } x = \frac{1 + n\ \text{coſ.}\,\alpha}{n + \text{coſ.}\,\alpha}.$$

SECOND SOLUTION, *by Mr.* J. COLLINS, *Kenſington.*

Put s = ſine of the arch α, and c its coſine, the radius being denoted by r. Alſo put x = the ſine of the arch φ, and y = ſine of the arch ψ.

Then by known theorems.

$$s \times \text{coſ.}\,\varphi = ry + cx$$
$$s \times \text{coſ.}\,\psi = rx + cy.$$

From whence

$$\frac{\text{coſ.}\,\varphi}{\text{coſ.}\,\psi} = \frac{ry + cx}{rx + cy} = \frac{r + \frac{cx}{y}}{\frac{rx}{y} + c} = \frac{r + nc}{rn + c}. \qquad Q.E.I.$$

Solutions were alſo received from Meſſrs. Bazley, Boole, Butterworth, Collins, Elliot, Myers, Pickering, Roville, *and* Simpſon.

XV. QUESTION 75, *by* NAUTICUS.

Required the rectification of the line which is the projection of a great circle paſſing through a given point of the equator, and making with it a given angle in Mercator's Chart?

SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

Retaining the ſame notation as in the ſolution to queſtion 46, in page 105 of the Repoſitory, let us put farther z for the length of the curve line whoſe rectification is required. Then from what has been ſhewn in the ſolution referred to

$$\left\{ = \sqrt{(\dot{x}^2 + \dot{y}^2)} \right\} = \sqrt{\left(\dot{\varphi}^2 + \frac{a^2\dot{\varphi}^2\,\text{coſ.}^2\,\varphi}{1 + a^2\,\text{ſin}^2.\,\varphi}\right)} = \frac{\sqrt{(1 + a^2)}\,\dot{\varphi}}{\sqrt{(1 + a^2 - a^2\,\text{coſ}^2\varphi)}}.$$

Hence, dividing both numerator and denominator by $\sqrt{(1+a^2)}$, and putting e for the ſine of the angle, which the great circle makes with the equator, that is, for $\frac{a}{\sqrt{(1+a^2)}}$, we find

$$\dot{z} = \frac{\dot{\varphi}}{\sqrt{(1 - e^2 \operatorname{coſ.}^2 \varphi)}}, \text{ or, taking } \theta = 90^\circ - \varphi,$$

$$\dot{z} = \frac{-\dot{\theta}}{\sqrt{(1 - e^2 \operatorname{ſin.}^2 \theta)}}.$$

This fluxion has ſeveral remarkable properties, which have been conſidered by Euler, Lagrange, and Legendre, particularly by the laſt mentioned Author in his *Memoire ſur les Tranſcendantes Elliptiques*. By a particular transformation its fluent admits of being expreſſed in a peculiar and very elegant manner. On this ſubject, ſee the laſt mentioned work, alſo a valuable paper, on the integration of certain differential expreſſions by Mr. Woodhouse of Cambridge, in the London Tranſactions for 1804. I ſhall only add, that the fluent may be expreſſed by two elliptic arches, the fluxion being the ſame as that which has occurred in the ſolution of Queſtion 16, in page 38 of this Vol. but it is proper to obſerve, that in the 11th and 17th lines of that page, the word ſemi-conjugate ſeems, in each, to have been put by miſtake for ſemi-tranſverſe.

XVI. QUESTION 76, *by* Theodosius.

Having given the baſe of a ſpherical triangle upon the ſurface of the ſphere and its area; required the locus of its vertex?

Solution, *by Mr.* Lowry, *R. M. College.*

It has been ſhewn in ſeveral places that the locus required is a ſmall circle of the ſphere, equal and parallel to a ſmall circle, deſcribed through the extremities of the given baſe; and the truth of this has generally been proved, by firſt ſhewing that ſpherical parallelograms on the ſame baſe, and between the ſame equal and parallel circles, are equal in area. This beautiful theorem is analogous to the thirty-fifth propoſition of the firſt book of Euclid, and is a remarkable inſtance of the analogy which frequently ſubſiſts between ſpherical and rectilineal figures. In what follows I ſhall determine the locus from the properties of the ſupplementary triangle, as this method is not only remarkably ſimple, but leads to the diſcovery of ſome other local theorems.

Let ABC (fig. 120, pl. 6.) be a ſpherical triangle deſcribed on the given baſe AB, and having its area, or which is the ſame, the ſum of its three angles equal to a given quantity. About the angular points A, B, C, as poles, deſcribe the ſupplementary triangle DEF, and continue the arcs DF, EF, till they meet as at G. Then it is well known that the ſides DF, EF, and DE, are the ſupplements of the angles B, A, C, reſpectively, and the angle DFE the ſupplement of the baſe AB; therefore when the baſe AB and the ſum of the three angles A, B, C, are given, the angle DFE, and the ſum of the three ſides DF, EF, ED (or the perimeter of the triangle DFE) are given.

Conceive a ſmall circle to touch the great circles DE, DG, and EG at Q, P, and H; let O be the pole of this circle, and join OP, OH, OQ, and QC. Then, by the property of tangents, the arc DQ = DP, arc EQ = EH, and arc PF = FH; therefore PF and FH are each equal to half the perimeter of the triangle DEF, that is, equal to a given arc; and therefore the points H, P, are given, and of conſequence the pole O, and the radius OQ (OH or OP) are given.

Alſo becauſe C is the pole of the arc DE, it is evident that the locus of the point C will be the locus of the pole of the arc DE, and ſince DE touches the circle DQH at Q, the arc OQ is perpendicular to DE; and, conſequently, OQ produced will paſs through the pole C, and QC will be a quadrant; therefore the arc OC, is equal to the arc OQ + a quadrant, that is equal to a given arc; and O is a given point: therefore the locus of the point C is the circle CI, whoſe pole is O, and radius OC.

Becauſe the arcs, HO, PO are perpendicular to GE, GD, they will paſs through the poles A, B, of thoſe circles, and therefore the arcs AO, BO, are each the complement of the arc OQ, or the ſupplement of the arc OC; and, conſequently, a circle deſcribed about the pole O to paſs through the points A, B, will be equal and parallel to the circle CI, and will be the locus of that point on the ſphere which is diametrically oppoſite to C.

Now ſince the great circle AB interſects one of the equal parallel circles at A and B, it muſt neceſſarily interſect the other parallel on the oppoſite ſide of the ſphere. Let this be at *m* and *n*; then, when C coincides with either of thoſe points, Q will be either at H or I, and the three ſides of the triangle ABC will coincide. Moreover, if C be beyond either of thoſe points, Q will then fall between H and I on the ſide next to C, and the perimeter will be a variable quantity, which is contrary to the hypotheſis, and therefore the locus is limited to that particular part of the circle mCn which is included between the points m and n.

We may infer from hence, that when the vertical angle and perimeter of a ſpherical triangle are given, the loci of the poles of the baſe will be two equal and parallel circles given in magnitude and poſition.

It is alſo obvious that the arcs Am, Bn, are each equal to the ſupplement of the baſe, and that the arc OF biſects the baſe AB at q

Again, if a ſpherical triangle be deſcribed on the given baſe AB, ſo that the difference between the vertical angle and the ſum of the angles at the baſe may be equal to a given angle, the locus of the vertex will be a circle.

Let the ſupplemental triangle, DEF (fig. 121, pl. 6.) be deſcribed as before, and let O be the pole of the circle PHQ inſcribed in the triangle, and join OQ, OP, OH, and OC; then, becauſe the baſe and the difference between the vertical angle and the ſum of the angles at the baſe are given, the angle DFE and the difference between the ſum of the ſides DF, EF, and the baſe DE will be given. Now the arc DP = DQ, arc EH = EQ, and arc FP = FH, therefore FH and FP are each equal to half the difference between the ſum of the ſides DF, EF, and the baſe DE, and therefore equal to a given arc. Hence it appears, in the ſame manner as before, that the point O and the arc OC are given; and, conſequently, the locus of the point C is a circle whoſe pole is O, and radius OC.

The arcs OH, OP, produced will paſs through the poles A and B reſpectively, and the arcs AO, BO, are each equal to the complement of the radius OQ, or equal to the arc OC, and therefore the locus required is the circle which circumſcribes the triangle ACB.

If the arcs AC, BC, be produced till they meet as at G, the locus of the point G will be a circle equal and parallel to that which is the locus of the point C, and will therefore paſs through the three points which are diametrically oppoſite to the points A, B, C. We may therefore infer, that if the vertical angle and the difference between the ſum of the ſides and baſe are given, the loci of the poles of the baſe are two given circles equal and parallel to each other.

The circle inſcribed in the triangle DEF, and the circle circumſcribing the triangle ABC have the ſame centre, and the radius of the one is the complement of the radius of the other.

That the circumſcribing circle is the locus of the vertex when the difference between the vertical angle and the ſum of the angles at the baſe is a conſtant quantity, may be proved in a manner different from the above, and ſuggeſts a very ſimple method of determining the locus, when the area is conſtant, without having recourſe to the properties of the ſupplementary triangle.

Let ABC (fig. 122, pl. 6.) be the triangle whoſe baſe and area are given, and continue the ſides CA, CB, till they meet at D, then it is evident that the angles D and C are equal, and $\angle\,\text{CAB} = 180^\circ - \angle\text{DAB}$, and $\angle\text{CBA} = 180^\circ - \angle\text{DBA}$; therefore the ſum of the angles A, C, B, is equal to $360^\circ - (\angle\text{DAB} + \angle\text{DBA}) + \angle\text{D} =$ to a given quantity S, or $\angle\text{DAB} + \angle\text{DBA} - \angle\text{D} = 360^\circ - \text{S}$.

Let O be the pole of a circle deſcribed through the points, A, B, D, and draw the radii OA, OB, OD; then, by reaſon of the iſoſceles triangles, $\angle\text{OBD} = \text{ODB}$, $\angle\text{OAD} = \text{ODA}$; therefore $\angle\text{D} = \text{OBD} + \text{OAD}$; and, conſequently, $\angle\text{DAB} + \text{DBA} - \angle\text{D} = \text{OAB} + \text{OBA} = 2\text{OAB} = 360^\circ - \text{S}$; therefore $\text{OAB} = 180^\circ - \tfrac{1}{2}\text{S} =$ the ſupplement of half the given ſum of the angles, and becauſe AB is given, it is obvious that the pole O and the radius OD (OA, or OB) are given; therefore the locus of the point D is the circle DAB.

And ſince the point C is always at the diſtance of a ſemicircle from D, it is evident that the locus of C is a circle equal and parallel to that which is the locus of D.

The celebrated Legendre, in the notes to his Eléments de Géometrie, alſo determines the locus to be a circle: but by principles very different from the above. His inveſtigation is as follows:

Let ABC (fig. 123, pl. 6.) be a ſpherical triangle deſcribed on the given baſe AB, and having a given ſurface $\text{A} + \text{B} + \text{C} - n = \text{S}$; n being $= 180^\circ$.

Let the arc IPK be drawn perpendicular to the baſe AB ſo as to biſect it at I, and take IP = to a quadrant; then P will be the pole of the arc AB, and the arc PC, deſcribed through P and C, will be perpendicular to AB. Put $\text{AB} = c$, $\text{ID} = p$, $\text{CD} = q$, $\text{AC} = b$, and $\text{BC} = a$; then $\text{AD} = p + \tfrac{1}{2}c$ and $\text{DB} = p - \tfrac{1}{2}c$. And from the right angled triangles ACD, BCD, we have

$\text{coſ.}\,a = \text{coſ.}\,q\ \text{coſ.}\,(p - \tfrac{1}{2}c)$ and $\text{coſ.}\,b = \text{coſ.}\,q\ \text{coſ.}\,(p + \tfrac{1}{2}c)$.

But it is well known that $\text{cot.}\,\tfrac{1}{2}\text{S} = \dfrac{\text{cot.}\,\frac{1}{2}a\ \text{cot.}\,\frac{1}{2}b\ \text{coſ.}\,c}{\text{ſin.}\,\text{C}}$,

and writing for coſ. c, its value, $\dfrac{\text{coſ.}\,c - \text{coſ}\,a\ \text{coſ.}\,b}{\text{ſin.}\,a\ \text{ſin.}\,b}$,

we have $\text{cot.}\,\tfrac{1}{2}\text{S} = \dfrac{1 + \text{coſ.}\,a + \text{coſ.}\,b + \text{coſ.}\,c}{\text{ſin.}\,a\ \text{ſin.}\,b\ \text{ſin.}\,\text{C}}$,

and by ſubſtituting in this formula the values $\text{coſ.}\,a\ \text{coſ.}\,b = 2\ \text{coſ.}$

q coſ. p coſ. $\frac{1}{2}c$, $1 +$ coſ. $c = 2$ coſ.^{2} $\frac{1}{2}c$, ſin. b ſin. C = ſin. c ſin. B = 2 ſin. $\frac{1}{2}c$ coſ. $\frac{1}{2}c$ ſin. B, we have

$$\text{cot.}\ \tfrac{1}{2}S = \frac{\text{coſ.}\ \frac{1}{2}c + \text{coſ.}\ p\ \text{coſ.}\ q}{\text{ſin.}\ a\ \text{ſin.}\ \frac{1}{2}c\ \text{ſin.}\ B}.$$

But in the right angled triangle BCD, ſin. a ſin. B = ſin. q, therefore cot. $\frac{1}{2}$S $= \dfrac{\text{coſ.}\ \frac{1}{2}c + \text{coſ.}\ p\ \text{coſ.}\ q}{\text{ſin.}\ \frac{1}{2}c\ \text{ſin.}\ q}$, or coſ. p coſ. q = cot. $\frac{1}{2}$S ſin. $\frac{1}{2}c$ ſin. q — coſ. $\frac{1}{2}c$, which is the relation between p and q, and ſerves to determine the line which is the locus of all the points C.

On IP produced take PK $= x$, and let the arc joining KC $=$ y, then in the triangle PKC we have PC $= \frac{1}{2}n - q$ and the angle KPC $= n - q$, and the ſide KC found by the proper formula is coſ. KC = coſ. KPC ſin. PK ſin. PC — ſin. x coſ. q coſ. p, in which ſubſtituting for coſ. p coſ. q its value cot. $\frac{1}{2}$S ſin. $\frac{1}{2}c$ ſin. q — coſ. $\frac{1}{2}c$ we ſhall have coſ. y = ſin. x coſ. $\frac{1}{2}x$ + ſin. q (coſ. x — ſin. x cot. $\frac{1}{2}$S ſin. $\frac{1}{2}c$); hence it appears that if we take (coſ. x — ſin. x cot $\frac{1}{2}$S ſin. $\frac{1}{2}c$) = 0, or which is the ſame, cot. x = cot. $\frac{1}{2}$S ſin. $\frac{1}{2}c$ we ſhall have coſ. y = ſin. x coſ. $\frac{1}{2}c$ and the value of y conſtant.

Therefore, if PK be taken ſuch that its cot. = cot. $\frac{1}{2}$S ſin. $\frac{1}{2}c$, the vertices of all the triangles deſcribed on the ſame baſe c, and having the same ſurface S, are in the circumference of a ſmall circle deſcribed from the point K, as a centre, with the polar diſtance CK, ſuch that coſ. KC = ſin. PK coſ. $\frac{1}{2}c$.

Legendre adds, that this beautiful theorem is due to Lexell, (See Tom. V. Part I. de *Nova Acta Petropolitana.*)

Meſſrs. Butterworth *and* Collins *ſent Solutions to this Queſtion.*

XVII. QUESTION 77, *by Mr.* JOHN WRIGHT, *Norley.*

Required the fluent of $\dot{x}\sqrt{\left[\frac{1+x^6}{(1+x^2)^5}\right]}$, with the aſſiſtance of the conic ſections ?

SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

In the firſt place,

$$\dot{x}\sqrt{\frac{1+x^6}{(1+x^2)^5}} = \frac{\dot{x}}{(1+x^2)^2}\sqrt{\frac{1+x^6}{1+x^2}} = \frac{\dot{x}}{(1+x^2)^2}\sqrt{(1-x^2+x^4)}.$$

Put $1 + x^2 = \frac{2x}{v}$; then $\frac{\dot{x}}{(1+x^2)^2} = \frac{v^2\dot{x}}{4x^2}$, and

$1 - x^2 + x^4 = (1+x^2)^2 - 3x^2 = \frac{4x^2}{v^2} - 3x^2 = \frac{4x^2}{v^2}(1 - \frac{3v^2}{4})$;

taking the roots, $\sqrt{(1 - x^2 + x^4)} = \frac{2x}{v}\sqrt{(1 - \frac{3v^2}{4})}$:

therefore $\frac{\dot{x}}{(1+x^2)^2}\sqrt{(1 - x^2 + x^4)} = \frac{v\dot{x}}{2x}\sqrt{(1 - \frac{3v^2}{4})}$.

By taking the fluxion of the equation $1 + x^2 = \frac{2x}{v}$, we have $2x\dot{x} = \frac{2v\dot{x} - 2x\dot{v}}{v^2}$ which gives $v\dot{x} = \frac{x\dot{v}}{1 - vx}$, or $\frac{v\dot{x}}{2x} = \frac{\frac{1}{2}\dot{v}}{1 - vx}$.

Alſo from the ſame equation we get $x = \frac{1 - \sqrt{(1 - v^2)}}{v}$, or $vx = 1 - \sqrt{(1 - v^2)}$; hence $1 - vx = \sqrt{(1 - v^2)}$, and therefore $\frac{v\dot{x}}{2x} = \frac{\frac{1}{2}\dot{v}}{1 - vx} = \frac{\frac{1}{2}\dot{v}}{\sqrt{(1 - v^2)}}$.

Conſequently,

$$\dot{x}\sqrt{\frac{1+x^6}{(1+x^2)^5}} = \frac{\dot{x}}{(1+x^2)^2}\sqrt{(1-x^2+x^4)} = \frac{v\dot{x}}{2x}\sqrt{(1-\frac{3v^2}{4^2})} = \tfrac{1}{2}\dot{v}\sqrt{\frac{1-\frac{3v^2}{4}}{1-v^2}},$$

which is at once recogniſed to be the fluxion of half the length of the arch of an ellipſis intercepted between the conjugate axis and a parallel ordinate at the diſtance of $v = \frac{2x}{1+x^2}$ from the centre; the tranſverſe axis of the ellipſe being 2, and conjugate 1.

SECOND SOLUTION, *by Mr.* THOMAS BAZLEY.

Let the given expreſſion be divided by $1 + x^2$ and it becomes $\dot{x}\sqrt{(\frac{1 - x^2 + x^4}{(1+x^2)^4})}$, or $\frac{\dot{x}}{(1+x^2)}\sqrt{(1 - \frac{3x^2}{(1+x^2)^2})}$. Now to reduce this to a form more convenient for integration, put $1 + x^2 = \frac{2}{1+z}$; then $\frac{\dot{x}}{1+x^2} = \frac{-\dot{z}}{2\sqrt{(1 - z^2)}}$, and by ſubſtitution the above expreſſion becomes, $\frac{-\dot{z}\sqrt{(1+3z^2)}}{4\sqrt{(1-z^2)}}$. The fluent of which is the fourth part of the elliptic arc, reckoned from the extremity of the conjugate axis, whoſe ordinate to the tranſverſe axe is z, ſemi-conjugate 1, and excentricity $\sqrt{3}$.

XVIII. QUESTION 78, *by* URSA MINOR.

It is required to demonſtrate, independently of fluxions, that when an arch of a circle is ſmall, twice the ſine + the tangent = three times the arch nearly?

FIRST SOLUTION, *by Mr.* IVORY, *R. M. College.*

LEMMA.

Let the ſmall arc ACB (fig. 124, pl. 6.) be biſected in C, and let AC be biſected in D: then the triangle ACB will be nearly equal to eight times the triangle ADC.

For, draw CO, DO to the centre of the circle, and let CO meet AB in m, and DO meet AC in n: then, becauſe the arch BA is ſmall the chord BA will be nearly double of the chord CA, and the chord CA nearly double of the chord DA: alſo, from the known property of the circle $Cm : Dn :: CA^2 : DA^2$ or nearly as $4 : 1$; therefore Cm is nearly quadruple of Dn. Now the triangle BCA : triangle CDA :: $BA \times Cm : CA \times Dn :: 2 \times 4 : 1 \times 1 :: 8 : 1$. .... *Q. E. D.*

DEMONSTRATION OF THE QUESTION.

Let AB (fig. 125, pl. 6.) be the ſmall arch, and AD its tangent: take the arch BE = arch AB, and join AE meeting BO drawn to the centre of the circle in C: biſect AB, BE in F and G, and join AF, FB, BG, GE: in like manner biſect the arches AF, FB, BG, GE, and inſcribe a figure of eight ſides in the ſegment ABE by drawing lines from the points of biſection to the points A, F, B, G, E: and ſuppoſe the inſcribing of figures, by means of continual biſections, to be continued ad infinitum.

Then, by the Lemma, the triangle EAB = 8 × triangle BFA; and the triangle BFA = 8 × triangle in the ſegment contained by the half of the arch FA; and ſo on, ad infinitum. But the circular ſegment ABE is compoſed of the $\triangle$ ABE + 2 × $\triangle$ AFB + 4 × $\triangle$ inſcribed in the ſegment AF + 8 × $\triangle$ inſcribed in the ſegment bounded by half the arch AF, and ſo on, ad infinitum: therefore the circular ſegment ABE

$$= \triangle \text{ ABE} \times \left(1 + \frac{2}{8} + \frac{4}{8^2} + \frac{8}{8^3} + \frac{16}{8^4} + \text{\&c. ad infinitum}\right)$$

$= \frac{4}{3} \times \triangle$ ABE*. Therefore the ſemi-ſegment ABC $= \frac{4}{3} \times \triangle$ ABC. But DO : OB :: OB : OC, therefore DB : BC :: OB : OC; and becauſe the arch AB is ſmall, therefore OB will differ little from OC; and, conſequently, DB = BC nearly. Therefore $\triangle$ ACD = 2 × $\triangle$ ABC: therefore ſemi-ſegment ACB $= \frac{2}{3} \times \triangle$ ACD.

Therefore the outward ſpace AFBD $= \frac{1}{3} \triangle$ ACD $= \frac{1}{2}$ ſemi-ſegment ABC, but the outward ſpace AFBD $= \frac{\text{AO}}{2}$ (AD − arch AB): and the ſemi-ſegment ABC $= \frac{\text{AO}}{2}$(arch AB − AC): therefore, arch AB − AC $= \frac{1}{2}$ (AD − arch AB): conſequently, arch AB $= \frac{2\text{AC} + \text{AD}}{3}$, or, to ſpeak in ſtrict geometrical language, the limit of the ratio of the ſmall arch AB to $\frac{2\text{AC} + \text{AD}}{3}$ is the ratio of equality, and, the ſmaller the arch AB is, the more nearly equal will it be to the quantity $\frac{2\text{AC} + \text{AD}}{3}$. *Q. E. I.*

SECOND SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

Let φ denote a ſmall arch of a circle, radius being unity.

$$\text{Then tan. } \varphi = \frac{\text{ſin. } \varphi}{\sqrt{(1 - \text{ſin}^2. \varphi)}} \left\{ \begin{array}{l} = \text{ſin. } \varphi \, (1 - \text{ſin}^2 \varphi)^{-\frac{1}{2}} \\ = \text{ſin.} \varphi + \frac{1}{2} \text{ſin}^3 . \varphi + \frac{3}{8} \text{ſin}^5 . \varphi + \text{\&c.} \end{array} \right.$$

Now when the arch φ is ſmall, the fifth and all higher powers of its ſine will be very ſmall, therefore in inveſtigating an approximation to the arch they may be neglected, thus we have

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* ARCHIMEDES has ſummed up this progreſſion, in his Quadrature of the Parabola.

$$\text{tan.}\,\varphi = \text{ſin.}\,\varphi + \tfrac{1}{2}\,\text{ſin.}^3\varphi \text{ nearly,}$$

$$\text{or, ſince ſin.}^3\varphi = \tfrac{3}{4}\,\text{ſin.}\,\varphi - \tfrac{1}{4}\,\text{ſin.}\,3\varphi,$$

$$\text{tan.}\,\varphi = \tfrac{11}{8}\,\text{ſin.}\,\varphi - \tfrac{1}{8}\,\text{ſin.}\,3\varphi \text{ nearly.}$$

Let us aſſume $\text{ſin.}\,\varphi = A\varphi + B\varphi^2 + C\varphi^3 + D\varphi^4 + E\varphi^5 + \&c.$ where A, B, C, D, &c. denote numbers which are independent of any particular value of φ, then, ſince by the arithmetic of ſines, when φ is negative its ſine is alſo negative, we have

$$-\,\text{ſin.}\,\varphi = -A\varphi + B\varphi^2 - C\varphi^3 + D\varphi^4 - E\varphi^5 + \&c.$$

and, ſubtracting the latter equation from the former, and dividing both ſides of the reſulting equation by 2,

$$\text{ſin.}\,\varphi = A\varphi + C\varphi^3 + E\varphi^5 + \&c.$$

hence it appears that B, D, &c., the coefficients of the even powers of φ, are each neceſſarily = 0, and, therefore, that the ſeries for the ſine muſt conſiſt only of the odd powers of the arch.

Let us divide both ſides of the ſeries by φ, and we have

$$\frac{\text{ſin.}\,\varphi}{\varphi} = A + C\varphi^2 + E\varphi^4 + \&c.$$

Now when $\varphi = 0$ all the terms of the ſeries vaniſh except A; hence it follows, that A expreſſes the limit of the ratio of the ſine of an arch to the arch itſelf, when that arch is ſuppoſed continually diminiſhed, but it may be rigidly demonſtrated independent of the doctrine of fluxions, that the ratio in queſtion is that of equality, therefore A = 1, and the ſeries for the ſine may be expreſſed thus,

$$\text{ſin.}\,\varphi = \varphi + B\varphi^3 + C\varphi^5 + \&c.$$

and ſubſtituting 3φ for φ

$$\text{ſin.}\,3\varphi = 3\varphi + 27B\varphi^3 + 243C\varphi^5 + \&c.$$

ſo that, rejecting the fifth and all the higher powers of the arch, for the ſame reaſon as we have already rejected the like powers of the ſine, we have

$$\text{ſin.}\,\varphi = \varphi + B\varphi^3 \text{ nearly,}$$

$$\text{ſin.}\,3\varphi = 3\varphi + 27B\varphi^3 \text{ nearly,}$$

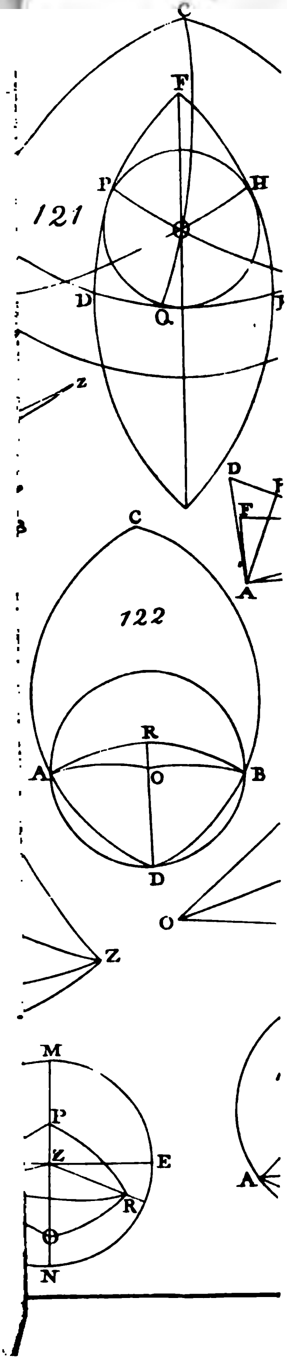
and ſince $\text{tan.}\,\varphi = \frac{11}{8}\,\text{ſin.}\,\varphi - \frac{1}{8}\,\text{ſin.}\,3\varphi$ nearly,

$$\text{tan.}\,\varphi = \varphi - 2B\varphi^3 \text{ nearly,}$$

hence, n being any number whatever,

$n\,\text{ſin.}\,\varphi + \text{tan.}\,\varphi = (n + 1)\,\varphi + (n - 2)\,B\varphi^3$ nearly.

Here if we ſuppoſe $n = 2$, the coefficient of φ^3 will be = 0, and thus $2\,\text{ſin.}\,\varphi + \text{tan.}\,\varphi = 3\varphi$ nearly. And as in obtaining this approximation we have rejected no quantities of a lower order than the fifth power of the ſine or of the arch, it may be inferred that the approximation is tolerably correct for ſmall arches.



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THIRD SOLUTION, *by Mr.* THOMAS BAZLEY.

It is demonſtrated by writers on trigonometry, that if a denotes the arc of a circle whoſe radius is unity, the correſponding

$$\text{ſine is } \ldots\ldots = a - \frac{a^3}{6} + \frac{a^5}{120} - \&c. = s$$

$$\text{and } 2 \times \text{ſine is} = 2a - \frac{a^3}{3} + \frac{a^5}{60} - \&c. = 2s$$

$$\text{and the tangent} = a + \frac{a^3}{3} + \frac{2a^5}{15} + \&c. = t$$

and if the arc be very ſmall all the terms after the ſecond may be omitted as being extremely ſmall in reſpect of the two firſt; hence by addition $2s + t = 3a$, which is the theorem in the queſtion?

XIX. QUESTION 79, *by Mr.* JOHN WRIGHT.

Given the difference of the angles at the baſe, and the length of a line drawn from the vertex to the middle of the baſe to conſtruct the triangle when the ſolid of the perimeter, the difference between the ſum of the ſides and the baſe, and the perpendicular let fall from the vertical angle upon the baſe is a maximum?

SOLUTION, *by Mr.* J. JOHNSON, *Birmingham.*

ANALYSIS. Suppoſe that ACB (fig. 126, pl. 7.) is the triangle required, IK the diameter of the circumſcribing circle perpendicular to and biſecting AB at D; join KC, IC and DC, and draw CP parallel to AB meeting IK in P: then there is given DC, the angle CKI, (half the difference of the angles at the baſe), or its complement the angle CIK, and the ſolid

$$(AC + CB + AB)(AC + CB - AB)\,DP = \left\{(AC + CB)^2 - AB^2\right\}$$

$\times$ DP, a maximum.

Now $(AC + CB)^2 = 4ID \times PK$ and $AB^2 = 4ID \times DK$, and their difference = 4ID × DP, therefore ID × DP² muſt be a maximum.

On DC let a ſemi-circle be deſcribed which will be the locus of the point P, alſo let a ſegment of a circle DCIL be deſcribed upon DC, to contain an angle equal to the angle DIC, or the complement of half the difference of the angles at the baſe, and this will be the locus of the point I. We have therefore only to draw the chord DI, ſo that ID × DP² may be a maximum. Let DH be the diameter of the circle DICL, then it is evident that both DI and DP will increaſe till DI coincides with DH, and therefore the point I cannot be on the ſame ſide of DH that the point I is, but muſt either be at H or between H and C. Now if it was at H the angle DCI would be equal to a right angle, and if between H and C greater than a right angle; that is, the angle DCI would be equal to or greater than the angle ICK, of which it is only a part, which is impoſſible. Therefore no maximum can properly take place, and of courſe the problem is unlimited.

If it was required to draw DI ſo that ID × DP² might be a maximum without having any regard to the triangle ABC. Let IC produced meet DB produced in R, and draw CQ parallel to DP; then it is manifeſt that DRC is a given angle, and therefore CQ (= DP) has to CR a given ratio, and DI to DR a given ratio; wherefore, when ID × DP² is a maximum, DR·CR² will be a maximum; wherefore in the triangle DCR, there are given the ſide DC, the oppoſite angle DRC and DR × CR² a maximum, which is the ſame as queſtion 842, Gentlemans' Diary.

Meſſrs. Simpſon *and* W. Smith *ſent ſolutions to this queſtion.*

XX. QUESTION 80, *by* Dr. O'RIORDAN.

To this Queſtion no ſatisfactory anſwer has been received.

XXI. QUESTION 81, *by Mr.* THOMAS BAZLEY, *Bolton.*

It is required to determine the figure of an orifice to be cut in the vertical ſide of a veſſel or dam, whoſe water is always of the ſame conſtant height; of ſuch a nature, that the quantity of water *iſſuing* below the ſhuttle ſhall be always as the height of the opening, and that when the whole orifice is open q cubic feet ſhall iſſue per ſecond; the top of the orifice being n, and the bottom m feet, below the ſurface?

FIRST SOLUTION, *by Mr.* BAZLEY, *the Propoſer.*

Let UR (fig. 127, pl. 7.) be the ſurface of the water, BoDErCB the orifice; oDSErIo the part below the ſhuttle; AKIS the axis biſecting all the ordinates oIr perpendicularly, the ordinates being parallel to the ſurface of the water UAR. Put AK $= n$, AS $= m$, AI $= x$, oIr $= y$, and $c = 5 \cdot 3$ agreeably to the experiments of Boſſut and others. Then the velocity of the effluent water at I, is $cx^{\frac{1}{2}}$, and conſequently the fluxion of the quantity iſſuing below the ſhuttle is, $-cy\dot{x}x^{\frac{1}{2}} = -a\dot{x}$ by the nature of the queſtion, where a is conſtant; hence $cyx^{\frac{1}{2}} = a$, or $y = \dfrac{a}{cx^{\frac{1}{2}}}$, the equation of the curve, which is of the hyperbolic kind. Now to determine a, let $\text{K} - ax$ repreſent the correct fluent of $-a\dot{x}$; and, per question, when $x = m$, $\text{K} - ax = \text{K} - am = 0$, ſo that $\text{K} = am$; and when $x = m$, $\text{K} - ax = am - an = q$, whence $a = \dfrac{q}{m-n} = \dfrac{q}{\text{KS}}$.

Moreover it appears that the general correct expreſſion for the quantity of effluent water is, $am - ax = \dfrac{q}{m-n}(m-x) = \dfrac{q}{\text{KS}} \times \text{SI}$.

SECOND SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

To make the conſideration of this queſtion as eaſy as poſſible, let us imagine the ordinates of the curve which bounds the orifice to be parallel to the horizon and perpendicular to the axis thereof. Alſo let us ſuppoſe the velocity of the effluent water at any depth, to be equal to that generated by gravity in an heavy body falling freely the ſame depth. Put $g = 16\frac{1}{12}$ feet, and let y denote the length of an ordinate at the depth x below the ſurface; alſo let l denote the length of the lowermoſt ordinate.

Then by the hypotheſis $2\surd(gx)$ will expreſs the velocity of the effluent water at the ordinate y, and $2\surd(gm)$ will expreſs the velocity of the iſſuing water at the bottom of the orifice or at the ordinate l. And as the quantity of water iſſuing below the ſhuttle *is to be always* proportional to the height of the opening, it is evident that the quantity iſſuing at every ordinate in equal times muſt be the ſame.

Now the quantity of water iſſuing at any ordinate in the ſame time is proportional to the rectangle or product of the ordinate and the velocity of the efflux; therefore $2y\sqrt{(gx)} = 2l\sqrt{(gm)}$ or $y^2x = l^2m$, which is an equation expreſſing the nature of the curve which bounds the orifice.

It has already been obſerved, that the quantity of water iſſuing at every ordinate in equal times muſt be the ſame, and conſequently the water iſſuing through the whole orifice in a ſecond when open, will be the ſame as would iſſue in the ſame time through an horizontal rectangular hole whoſe ſides are $2l$ and $m - n$ at the depth m below the ſurface.

Now the quantity of water which would iſſue through an horizontal rectangular hole whoſe ſides are $2l$ and $m - n$ at the depth m below the ſurface of the water in a ſecond of time will, by the known principles of hydroſtatics, be expreſſed by $4l\,(m-n)\sqrt{(gm)}$, and this by the queſtion is to be equal to q cubic feet, whence we get

$$l = \frac{q}{4(m-n)\sqrt{(mg)}} \text{ and } y^2x = l^2m = \frac{q^2}{16(m-n)^2g}.$$

Mr. Elliott *ſent a Solution to this queſtion.*

XXII. QUESTION 82, *by Mr.* W. WALLACE.

$$\text{Given } \left\{ \begin{array}{l} \text{coſ. } \varphi + \text{coſ. } \psi = a \\ \text{coſ. } 5\varphi + \text{coſ. } 5\psi = b \end{array} \right\} \text{To find coſ. } \varphi \text{ and coſ. } \psi\,?$$

FIRST SOLUTION, *by the Propoſer.*

Let coſ. φ and coſ. ψ be denoted by x and y reſpectively, and let rad. $= 1$, then by the arithmetic of ſines

$$\text{coſ. } 5\varphi = 16x^5 - 20x^3 + 5x,$$
$$\text{coſ. } 5\psi = 16y^5 - 20y^3 + 5y;$$

and therefore,

$$\text{coſ. } 5\varphi + \text{coſ. } 5\psi = 16(x^5 + y^5) - 20(x^3 + y^3) + 5(x+y).$$

Now if we put a for $x+y$ and v for xy, it is evident that

$$x^5 + y^5 = a^5 - 5v(x^3 + y^3) - 10av^2,$$
$$x^3 + y^3 = a^3 - 3av.$$

Let theſe values of x^5+y^5, x^3+y^3 and $x+y$ be ſubſtituted in the above expreſſion for coſ. 5φ + coſ. 5ψ, which by hypotheſis is equal to a given number b, and we have

$$b = 16a^5 - 20a^3 + 5a + (60a - 80a^3)v + 80av^2.$$

The value of v, or of xy, may now be found by the reſolution of this quadratic equation, and hence, and from the equation $x+y=a$, the values of x and y, or of coſ. φ and coſ. ψ, may be found by the reſolution of another quadratic equation.

Note. In like manner if beſides coſ. φ + coſ. ψ there be given any one of theſe coſ. 2φ + coſ. 2ψ, coſ. 3φ + coſ. 3ψ, coſ. 4φ + coſ. 4ψ, the values of coſ. φ and coſ. ψ may alſo be found by the reſolution of quadratic equations: But if we advance to coſ. 6φ + coſ. 6ψ, the equation to be reſolved is a cubic.

SECOND SOLUTION, *by Mr.* G. WILLIAMS, *Caermarthen.*

Put $\frac{1}{2}y$ = coſ. φ, and $\frac{1}{2}v$ = coſ. ψ. Then, by the queſtion $y+v=2a$, and by Simpſon's Algebra, p. 200, we have

coſ. $5\varphi = \frac{y^5 - 5y^3 + 5y}{2}$ and coſ. $5\psi = \frac{v^5 - 5v^3 + 5v}{2}$.

Therefore $y^5 + v^5 - 5y^3 \quad 5v^3 + 5y + 5v = 2b$,

or becauſe $5y + 5v = 10a$,

$$y^5 + v^5 - 5y^3 - 5v^3 = 2b - 10a.$$

Aſſume $y - v = 2z$, then $y = a + z$, and $v = a - z$:

Therefore $y^5+v^5=(a+z)^5+(a-z)^5=2a^5+20a^3z^2+10az^4$

And $5y^3+5v^3 = 5(a+z)^3+5(a-z)^3 = 10a^3+30az^2$.

Hence, by tranſpoſition, &c.

$$z^4 + (2a^2-3)\,z^2 = a^2 - \tfrac{1}{5}a^4 + \frac{b}{5a} - 1, \text{ from which}$$

$z = \sqrt{\left\{\sqrt{(p^2+q)} - p\right\}}$, where $p = \frac{2a^2-3}{2}$, and $q = a^2 - \frac{1}{5}a^4 + \frac{b}{5a} - 1$.

Then coſ. $\varphi = \frac{1}{2}y = \frac{a+z}{2}$ and coſ. $\psi = \frac{1}{2}v = \frac{a-z}{2}$.

This Queſtion was anſwered by Meſſrs. Collins, Myers, *and* Pickering.

XXIII. QUESTION 83, *by Mr.* IVORY, *R. M. College.*

Let A and B be two points without a circle, of which O is the centre, and let $BO \times OC =$ ſquare of radius, and $AO \times OD =$ ſquare of radius; and let BD and AC meet in E. Then, if the chord FG be drawn through E, and AF, FB, BG, AG be joined, $AF \times FB : AG \times GB :: FE : EG$. Required the demonſtration?

FIRST SOLUTION, *by Mr.* JOHN JOHNSON, *Birmingham.*

Fig. 128, pl. 7. Draw the radii OF, OG, and produce OE to H ſo that $OE \times OH$ may be equal to the ſquare of the radius OF, and join FH: then becauſe $OE : OF :: OF : OH$, and the angle at O common to the triangles FOH, FOE, they are ſimilar, and conſequently the angle $OFG = OHF$; but the angle OFG is = to OGF (becauſe $OF = OG$); therefore the angle $OHF = OGF$, and the points O, G, H, F, are in a circle; wherefore $FE\cdot EG = OE\cdot EH$. Again, becauſe $OE\cdot OH = OD\cdot OA = OC\cdot OB$ (each being equal to the ſquare of the radius), the points A, D, E, H, are in a circle, and ſo are alſo the points A, D, C, B; therefore the $\angle AHO = ODE = OCA$, and conſequently the points A, H, C, O, are in a circle; and therefore $AE\cdot EC = OE\cdot EH$: but it has been ſhewn above that $FE\cdot EG = OE\cdot EH$; therefore $AE\cdot EC = FE\cdot EG$, and the points A, F, C, G, are in a circle.

Now ſince $OC\cdot CB =$ the ſquare of the radius, it is well known as a propoſition of the *Loci Plani* that

$$BF : CF :: BG : CG, \text{ or } BF : BG :: CF : CG;$$

$$\text{Therefore } AF\cdot BF : BG\cdot AG :: CF\cdot AF : CG\cdot AG.$$

But by the property of the circle and inſcribed trapezium

$$CF\cdot AF : CG\cdot AG :: EF : EG;$$

$$\text{therefore } AFBF : AG\cdot BG :: EF : EG.$$

Q. E. D.

COR. 1. $AF\cdot BF : AG\cdot BG :: FH : HG.$

COR. 2. $CF\cdot DF : DG\cdot CG :: EF : EG :: FH : HG.$

SECOND SOLUTION, *by Mr.* WM. SIMPSON, *Bolton.*

Fig. 128, pl. 7. Join FC, CG, and draw FP, GQ perpendicular to AC. Let OE be produced to meet the circle in I, and draw AH meeting OE in H, ſo that the angle EAH may be equal to the angle EOC.

Then becauſe the rectangles $CO \cdot OB$, $DO \cdot OA$ are each equal to the ſquare of the radius, the points C, D, A, B, are in a circle, and therefore the angles at C and D are equal, and conſequently the angle $AHE = ECO = EDO$, and therefore the points D, E, H, A, are in a circle.

Wherefore $DO \cdot OA$, or $(OI)^2 = EO \cdot OH = (OE)^2 + OE \cdot EH$; but the points A, H, C, O, are alſo in a circle; therefore $OE \cdot EH = AE \cdot EC$, and $(OI)^2 = (OE)^2 + AE \cdot EC$, or $(OI)^2 - (OE)^2 = AE \cdot EC$. But $(OI)^2 - (OE)^2$ is $= EK \cdot EI$, wherefore $AE \cdot EC = FE \cdot EG$, and the points A, F, C, G, are in a circle.

Now becauſe $OC \cdot OB$ is equal to the ſquare of the radius $FB : FC :: BG : CG$ (Playfair's Euclid, B. VI. P. F.),

and by permutation $FB : BG :: FC : CG$; therefore

$$AF \cdot FB : AG \cdot BG :: AF \cdot FC : AG \cdot CG.$$

But it has been ſhewn that the points A, F, C, G, are in a circle, and (Prop. C. Ibid) the rectangles under the diameter of this circle and the perpendiculars FP, CQ, are reſpectively equal to the rectangles $AF \cdot CF$, $AG \cdot CG$; therefore it is evident that

$AF \cdot FB : AG \cdot BG\ (:: FP : CQ) :: FE : EG$ (by ſimilar triangles).

Ingenious demonſtrations were alſo received from Meſſrs. Bazley, Limenus, *and* W. Smith.

XXIV. QUESTION 84, *by Mr.* BENJAMIN BEVAN.

To this Queſtion no anſwer has been received.

XXV. QUESTION 85, *by Dr.* O'RIORDAN.

To this Queſtion no ſatisfactory anſwer has been received.

XXVI. QUESTION 86, *by* SCOTICUS.

If four ſtraight lines interſect each other, and form four triangles, circles deſcribed about each of theſe triangles will paſs through one and the ſame point. Required the demonſtration?

FIRST SOLUTION, *by Mr.* GEORGE PICKERING.

Let AB, AC, GH, GC, (fig. 129, pl. 7.) be the four lines forming the four triangles AHE, ABC, GBE, and GHC. Let the circles AHE, EBG, circumſcribing the triangles AHE, EBG interſect in P. Then the circles ACB, HCG deſcribed about the remaining triangles ABC, GHC, will alſo paſs through the points P. Join PA, PH, PE, PB, and PG. Becauſe the point P, E, B, G, are in a circle, the angle PGB is equal to the angle PEA; but the angle PEA is equal to the angle PHA; therefore the angle PGB or PGC is equal to the angle PHA: hence the points P, H, C, G, are in the circumference of the ſame circle.

Again the angle PAH is equal to the angle PEG, that is equal to the angle PBG; conſequently the points P, A, C, B, are in a circle. Hence all the circles paſs through the same point P.

SECOND SOLUTION, *by Mr.* JOHN CAVILL, *Beighton.*

Let AF, AR, BF, and RC, (fig. 130, pl. 7.) be the four lines interſecting each other in the points A, B, F, R, E, and G, and forming thereby the four triangles ABF, ARE, BGR, and EGF; about the triangles BGR and EGF, let circles be deſcribed interſecting again in D, draw ED, FD, BD, and RD; then the angle EGF = EDF = BGR = DBR, that is the angle EDF = BDR, add the angle EDB to each, and we have the angle FDB = EDR; alſo the angle DEG = DFG, conſequently the triangles EDR, FDB are ſimilar, and ED : FD :: DR : DB; therefore the triangles BDR, FDC are ſimilar, and the angle DRB = FED.

Again the angle AED = EFD + EDF and AED + ARD = EFD + EDF + FED = two right angles; conſequently a circle will paſs through the points A, E, D, R. And ſince the angle EDR = FDB, and the angle EAR + EDR (= two right angles) = FAB + FDB, therefore a circle will paſs through the points A, F, D, B. Therefore circles deſcribed about the four triangles AFB, AER, BGR, and FEG, will all paſs through the ſame point D, which was to be demonſtrated.

Meſſrs. Bazley, Collins, Myers, Roville, Simpſon, W. Smith, *and* Williams, *anſwered it.*

XXVII.

XXVII. QUESTION 87, *by Mr.* JAMES NORTH.

From the three angular points of a given plane triangle to draw three right lines meeting in the circumference of a given circle ſo as to have a given ratio to one another, the plane of the circle being perpendicular to the plane of the triangle.

SOLUTION, *by Mr.* LOWRY, *R. M. College.*

Let ABC (fig. 131, pl. 7.) be the given triangle, and P a point above or below the plane ABC, ſuch, that the lines AP, BP, CP, drawn from that point to the angular points of the triangle may have a given ratio.

Take the points E, F, and G, H, ſuch, that AE may be to EB and AF to BF in the given ratio of AP to BP; and that AG may be to GC and AH to CH in the given ratio of AP to CP.

Let two circles be deſcribed on the diameters EF, GH, interſecting each other in the points D, *d*, and join AD, A*d*, BD, B*d*, CD and C*d*; then it is well known (Apol. Loci Plani, Lib. 2, Prop. 2.) that if two lines be drawn from the points A,B, to any point in the circumference of the circle deſcribed on EF, they will have the ſame ratio as AE to EB, or AP to BP; and alſo that two lines drawn from the points A, C, to any point in the circumference of the circle deſcribed on GH, will have the ſame ratio as AG to GC, or AP to CP; therefore it is obvious that the lines AD, BD, CD, and A*d*, B*d*, C*d*, have the ſame ratio as the lines AP, BP, CP.

Now if the points D, *d*, coincide, or the circles only touch each other, it is manifeſt that the point of contact will be the only point from which lines can be drawn to the points A, B, C, ſo as to have the given ratio. But when the circles interſect each other as at D and *d*, let a circle be deſcribed on the diameter D*d*, and have its plane perpendicular to the plane ABC, then the point required may be any where in the circumference of this circle.

For let P be any point in the circumference, O the centre, and I the centre of the circle EDF. Join DI, OI, PI, and PO; then becauſe D*d* is biſected in O, and the plane DP*d* is perpendicular to ACB, OI, and OP, are both perpendicular to OD; and OP is equal to OD, and OI common to both the triangles IOD, IOP; therefore, IP is equal to ID = IE = IF; and, conſequently, the point P is in the circumference of a circle

whole

whoſe diameter is EF; and, therefore, *(Ibid.)* AP will be to BP in the given ratio of AE to EB; and, in like manner, it is ſhewn that AP is to CP in the given ratio of AG to GC; therefore the lines drawn from the angular points A, B, C, to any point whatever in the circumference DP*d* will be in the given ratio.

It therefore appears that the poſition of the point P can only be determined when the circumference of the circle DP*d* happens to meet the circumference of the circle which is given by poſition, for in any other caſe the problem will be impoſſible.

If the diſtances AP, BP, CP are equal, the locus will be a ſtraight line perpendicular to the plane of the triangle ABC, and paſſing through the centre of its circumſcribing circle.

What has been ſaid will appear very evident if we conceive the two circles EDF, GDH, to revolve round their diameters ſo as to generate two ſpherical ſuperficies, for the mutual interſection of thoſe ſuperficies will be the circle DP*d*, which is the locus of the point P.

We may alſo from hence determine the poſition of a point ſuch, that if lines be drawn from it to the four angular points of a given trapezium, they may have a given ratio.

XXVIII. QUESTION 88, *by Mr.* JAMES IVORY.

Let there be an ellipſe of which AB is the greater, and ED the leſs axis and centre C. Take CH in either axis equal to the ſum or difference of the two ſemi-axis CB and CD, and upon CH let a circle be deſcribed; through B, the extremity of the axis, draw a ſtraight line to cut the circle in F and G, and through F and G draw two diameters of the ellipſe PC and QC. Then, the line FG will be equal to the ſum of the ſemi-diameters CQ and CP in the one caſe, and to their difference in the other; and if any line LK, equal to FG, be interpoſed between the diameters CP and CQ, and in LK (or in LK produced beyond the point K when CH is equal to the difference of the two axis,) LM be taken equal to CP, the point M will be in the periphery of the ellipſe. Required a demonſtration?

FIRST SOLUTION, *by the Proposer.*

This queſtion will be beſt demonſtrated by premiſing the following propoſitions.

PROPOSITION I. THEOREM, Fig. 132, 133, pl. 7.

Having deſcribed a circle on CH, the ſum or difference of the axis of an ellipſe as in the queſtion, and having drawn CF to the extremity of any chord paſſing through B; then CP, the ſemi-diameter of the ellipſe, will be equal to BG.

Join CG, CH; draw BR parallel to CG, BS parallel to GH, and PN perpendicular to AB. Then, on account of ſimilar triangles,

$$CB^2 : BH^2 :: CS^2 : BR^2.$$

But $BH = CD$, and from the nature of the ellipſe

$$CB^2 : CD^2 \text{ or } BH^2 :: AN \times NB : NP^2.$$

$$\text{Therefore } CS^2 : BR^2 :: AN \times NB : NP^2$$

$$\text{and, by alternation } CS^2 : AN \times NB :: BR^2 : NP^2.$$

Becauſe the angle CGH is a right angle, therefore BR, parallel to CG, is perpendicular to GH. And becauſe the angles PCN, BGR, in the ſame ſegment are equal to one another, therefore the right angled triangles PCN and BGR are ſimilar:

$$\text{Therefore } GR^2 : CN^2 :: BR^2 : PN^2.$$

Conſequently, by what was before proved,

$$CS^2 : AN \times NB :: GR^2 \text{ or } BS^2 : CN^2$$

$$\text{and, by alternation, } CS^2 : BS^2 :: AN \times NB : CN^2$$

$$\text{by compoſition, } BC^2 : BS^2 :: BC^2 : CN^2$$

Therefore $BS = GR = CN$. Therefore the two triangles CPN, BGR, are equal in all reſpects, and $PC = BG$.

Q. E. D.

Corol. If, in the line CF, CP be taken equal to BG, the point P will be in the circumference of the ellipſe.

PROPOSITION II. LEMMA, Fig. 134, Pl. 7.

Let the two parts of the line AB, be reſpectively equal to the two parts of the line EG, *viz.* $AC = EF$ and $BC = FG$; and further, let the angle ADC be equal to EHF, and the angle CDB to FHG: then will CD be equal to FH.

Upon AC, EF, let two ſegments of circles be deſcribed, each to contain an angle equal to the two equal angles ADC, EHF; and, in like manner, upon CB and FG, let ſegments be deſcribed, each to contain an angle equal to the two equal angles CDB and FHG. Then it is obvious that the circumferences of the two ſegments upon AC, CB will interſect one another in C and D; and the two circumferences of the ſegments on EF, FG will interſect one another in F and H. Now let the figure upon EF be applied to the figure on AB, ſo that E may be on A, and the line EG on the line AB; then will F be on C, and G on B; and the circumference of the ſegment on EF will coincide with the circumference of the ſegment on AB (Eu. 24, III). And in like manner the circumference of the ſegment on CD will coincide with the circumference of the ſegment on FG: therefore ſince F coincides with C, H neceſſarily coincides with D (Eu. 10, III). Therefore FH coincides with CD, and is equal to it.

Q. E. D.

DEMONSTRATION OF THE QUESTION. Fig. 132, 133, Pl. 7.

By Prop. 1, GB is equal to CP, and BF to CQ, therefore FG is equal to the ſum of CP and CQ in the one caſe, and to the difference of CP and CQ in the other caſe.

Join CM, meeting the circumference again in F; draw the chord FBU and join FU, GU. Then it is eaſy to demonſtrate (21 and 22, Eu. III.) that the angle FUB is equal to the angle MCK, and the angle BUG to MCL: it is alſo clear that LM = CP = BG, and MK = BF, therefore, by the Lemma CM = BU. Therefore (Cor. Prop. 1.) the point M is in the circumference of the ellipſe.

SCHOLIUM.

Prop. 1st. furniſhes a good method of deſcribing an ellipſe by finding an indefinite number of points in the periphery. For any chords as FG and TU, being drawn through B, if CF and CT be drawn, and CP, C*p* be taken each equal to BG, and CM, C*m* each equal to BU: then P, *p*, M, and *m* will be points in the periphery of the ellipſe.

SECOND SOLUTION, *by Mr.* LOWRY.

Fig. 135, 136, pl. 7. From Q, to the tranſverſe axis AB, apply QR equal to the ſemi-conjugate axis CD, and produce it to meet DE in I; then it is proved by moſt writers on conic ſections, (and is, in fact, the property on which the elliptical *compaſſes are conſtructed*, ſee Robertſon's Conics, Prop. 24, B II.)

11.) that IR is equal to the ſum or difference of the ſemi-axis CB, CD, that is, equal to CH. Join GH, FH, and FC; then the angles CFH, ICR, being right angles, and the angles HFB, HCQ equal (Eu. 21 and 22. 111.), and CH = IR and CB = IQ, the triangles BCF, QCI will be equal in every reſpect as may be very eaſily proved; and, conſequently, BF is equal to CQ. It may be proved in a like manner that BG is equal to CP, and therefore FG is equal to the ſum or difference of CP and CQ.

Again, let a circle be deſcribed through the points C, L, K, cutting CB in O, and CD in T, and join TO, OM, OK, and LK. Then becauſe FG and LK are equal, and ſubtend the ſame angle at C, the circles CGFH, CKL will be equal; and the angles FCH, OTK, on the ſame arc OK, are equal. And the angles OTL, HCG, on the arc OL are equal, therefore the angles OKM, HFB, are equal (Eu. 21 and 22, 111.); and the triangles OKM, HFB, having the ſide OK = HF, the ſide KM = FB or CQ, and the angle OKM = HFB, will have OM = HB = CD, and the angle KOM = FHB. Alſo the angle TOK is equal to the angle CHF, each being the complement of the angle OCF, and therefore the angles KOM, TOK, together are equal to two right angles, and the points T, O, M, are in a ſtraight line. But ſince TCO is a right angle, TO is the diameter of the circle CKL, and equal to CH; and, conſequently, TM and OM are reſpectively equal to the axis CB, CD, and therefore the point M is in the ellipſe by the propoſition referred to above.

It is evident from what has been already done, that the firſt part of the propoſition is true when AB, DE, are any two diameters whatever, if inſtead of a circle being deſcribed on CH, as a diameter, a ſegment CGFH be deſcribed to contain an angle CFH equal to the angle ECB included between the diameters.

It is alſo obvious, from hence, how the axis may be found when two diameters of an ellipſe are given. For if LK be interpoſed between the diameters CP, CQ, and equal to their ſum or difference, and KM be taken equal to CQ, and a circle be deſcribed through the points C, L, K; then if MT be drawn through the centre of the circle meeting the circumference in O and T, MO and MT will be equal to the ſemi-axis, and their poſition will be determined by drawing CT and CO, and taking CB = MT, and CD = MO.

Ingenious Solutions were alſo received from Meſſrs. Bazley, Cavill, Simpſon, *and* W. Smith.

XXIX. QUESTION 89, *by* PHILOMATHES OXONIENSIS.

Required the fluent of $\frac{\dot{x}\sqrt[3]{x}}{(1+\sqrt{x})^3}$, generated while x increaſes from 0 to 2; and alſo the fluent of $\frac{\dot{x}x^3}{x^3-1)^{\frac{7}{3}}}$, generated while x increaſes from $\frac{11}{10}$ to 2?

FIRST SOLUTION, *by* J. H.

This queſtion contains two problems, the firſt of which may be ſolved thus:

Put $v^6 = x$; then $\sqrt{x} = v^3$, $\sqrt[3]{x} = v^2$, and $\dot{x} = 6\dot{v}v^5$, and the given expreſſion becomes $\frac{6\dot{v}\,v^7}{(1+v^3)^3}$, the fluent of which is obviouſly $= \frac{6v^8}{8(1+v^3)^3} + \frac{6\cdot 9\,v^{11}}{8\cdot 11(1+v^3)^4} + \frac{6\cdot 9\cdot 12\,v^{14}}{8\cdot 11\cdot 14(1+v^3)^5} + \frac{6\cdot 9\cdot 12\cdot 15\,v^{17}}{8\cdot 11\cdot 14\cdot 17(1+v^3)^6}$, &c. *ad infinitum*, which ſeries will vaniſh with v, and therefore needs no correction, and, when $v^6 = 2$, it will converge by the powers of $\frac{\sqrt{2}}{1+\sqrt{2}} = \frac{7}{12}$ nearly, ſo that its value may be obtained by computing a moderate number of terms. It may, however, be transformed into other ſeries which ſhall converge much more ſwiftly. (See the *Philoſophical Tranſactions* for the year 1798, Part I. Art. IX.) Or, the fluent of the above expreſſion may be computed by a deſcending ſeries which ſhall converge by the powers of $\frac{1}{1+\sqrt{2}} = \frac{5}{12}$ nearly. But, ſince the fluent is eaſily attainable by logarithms and circular arches, which are ſeries already computed, I ſhall now treat of that method.

It is eaſy to perceive that the fluent of $\frac{6\dot{v}v^7}{(1+v^3)^3}$ may be taken in this form, viz. $\frac{Av^5}{(1+v^3)} + \frac{Bvv}{1+v^3} + \int\frac{C\dot{v}v}{1+v^3}$, in which the coefficients A, B, and C, may be computed with great facility. But a better form for numerical calculation may be derived from Madam *Agneſi's* Theorem in her *Analytical Inſtitutions*, Book III. Sect. I. Art. 56; obſerving, firſt, that the indices of x, in the numerator of the fraction which ſhe aſſumes, will always be an arithmetical progreſſion decreaſing by the conſtant difference m: And, ſecondly, that when n (ſtill uſing her notation) is greater than m, let m be taken from n as often as it may, which call p. times, and put the remainder, $n - pm = r$; then will

$$\int\frac{\dot{x}x^n}{(x^m \pm a^m)^u} = \frac{Bx^{r+um-2m+1} + Cx^{r+um-3m+1} + Dx^{r+um-4m+1} + \&c.}{(x^m \pm a^m)^{u-1}}$$

$+\int\frac{A\dot{x}x^r}{x^m \pm a^m}$, from the fluxion of which equation the values of A, B, C, D, &c. will eaſily be found, and then no more remains to be done than to compute the fluent of the term $\frac{A\dot{x}x^r}{x^m \pm a^m}$ by the well known method of logarithms and circular arches.

Now, by means of this equation we ſhall find

$$\int\frac{6\dot{v}v^7}{(1+v^3)^3} \text{ to be } = -\frac{8v^5+5v^2}{3(1 \pm v^3)^2} + \int\frac{10\dot{v}v}{3(1+v^3)}\text{; and}$$

ſince the correct fluent of

$$\frac{v\dot{v}}{1+v^3} = \tfrac{1}{3}\text{h.l.}\frac{\sqrt{(1-v+vv)}}{1+v} + \tfrac{2}{3} \times \left\{\begin{matrix}\text{cir. arch}\\ +\text{cir. a.}\end{matrix}\right\}\text{to r.}\frac{\sqrt{3}}{2}\ \&\ \left\{\begin{matrix}\text{tang.}\ v-\tfrac{1}{2}\\ \text{tang.}+\tfrac{1}{2},\end{matrix}\right.$$

we have the correct

$$\int\frac{6\dot{v}v^7}{(1+v^3)^3} = -\frac{8v^5+5v^2}{3(1+v^3)^2} + \tfrac{10}{9}\text{ h. l. }\frac{\sqrt{(1-v+vv)}}{1+v}$$

$$+\tfrac{20}{9} \times \left\{\begin{matrix}\text{circ. arch}\\ +\text{ circ. a.}\end{matrix}\right\}\text{to rad. }\frac{\sqrt{3}}{2}\ \&\ \left\{\begin{matrix}\text{tang.}\ v-\tfrac{1}{2}\\ \text{tang.}+\tfrac{1}{2}.\end{matrix}\right.$$

And when $v^6 = 2$, theſe terms, expreſſed in numbers (according to my calculation) are as below.

$$\begin{array}{rcl} -\dfrac{8v^5+5v^2}{3(1+v^3)^2} & = & -\ 1{\cdot}175502 \\ \dfrac{10}{9}\text{ h.l. }\dfrac{\sqrt{(1-v+vv)}}{1+v} & = & -\ 0{\cdot}764642 \\ \hline & & -\ 1{\cdot}940144 \\ \dfrac{20}{9} \times \left\{\begin{matrix}\text{the two}\\ \text{circ. arches}\end{matrix}\right\} & = & +\ 2{\cdot}207023 \\ \hline & & +\ 0{\cdot}266879 = \text{the value required.} \end{array}$$

A ſolution of the ſecond problem contained in this queſtion, in algebraic terms, is attainable by Art. 61 of the 1ſt Sect. of the book above referred to, thus:

Put $x^3 - 1 = zx^3$; then, $x^3 = \frac{1}{1-z}$, $x = \frac{1}{(1-z)^{\frac{1}{3}}}$

$\dot{x} = \frac{\dot{z}}{3(1-z)^{\frac{4}{3}}}$, $x^3 - 1 = \frac{z}{1-z}$, and $(x^3-1)^{\frac{7}{3}} = \frac{z^{\frac{7}{3}}}{(1-z)^{\frac{7}{3}}}$,

by ſubſtitution, $\frac{\dot{x}x^3}{(x^3-1)^{\frac{7}{3}}} = \frac{\dot{z}}{3(1-z)^{\frac{4}{3}}} \times \frac{1}{1-z} \times \frac{(1-z)^{\frac{7}{3}}}{z^{\frac{7}{3}}} =$

$\frac{\dot{z}}{3z^{\frac{7}{3}}}$, the fluent of which is $-\frac{1}{4z^{\frac{4}{3}}}$; and ſince $z = \frac{x^3-1}{x^3}$, the fluent is $-\frac{1}{4}\left(\frac{x^3}{x^3-1}\right)^{\frac{4}{3}} = -\frac{x^4}{4(x^3-1)^{\frac{4}{3}}}$. But, when $x = \frac{11}{10}$, the fluent ought to be $= 0$; therefore the correct fluent is

$+\frac{1}{4}\left(\frac{1331}{331}\right)^{\frac{4}{3}} - \frac{x^4}{4(x^3-1)^{\frac{4}{3}}}$; which, when x becomes $= 2$, will be $= \frac{1}{4}\left(\frac{1331}{331}\right)^{\frac{4}{3}} - \frac{4}{(7)^{\frac{4}{3}}} = 1{\cdot}299882$ the value required.

SECOND SOLUTION, *by* LIMENUS, *Bruton.*

Beginning with the firſt expreſſion, put $v^6 = x$, and then $\frac{\dot{x}\sqrt[3]{x}}{(1+\sqrt{x})^3} = \frac{6v^7\dot{v}}{(1+v^3)^3}$. Aſſume

$A \times$ fluent of $\frac{v\dot{v}}{1+v^3} - B\frac{v^2}{1+v^3} - C\frac{v^5}{(1+v^3)^2} =$ fluent of $\frac{6v^7\dot{v}}{(1+v^3)^3}$;

taking the fluxions and dividing by $\dot{v}$,

$A\,\frac{v}{1+v^3} + B\,\frac{v^4-2v}{(1+v^3)^2} + C\,\frac{v^7-5v^4}{(1+v^3)^3} = \frac{6v^7}{(1+v^3)^3}$;

multiplying by $(1+v^3)^3$,

$Av\,(1+v^3)^2 + B\,(v^4-2v)(1+v) + C\,(v^7-5v^4) = 6v^7$;

expanding the expreſſion and collecting the homologous terms,

$v^7\,(A+B+C-6) + v^4\,(2A-B-5C) + v\,(A-2B) = 0$, and by making the coefficients of the homologous terms $= 0$, we get

A =

$A=2B$; $C=\frac{2A-B}{5}=\frac{3B}{5}$; $A+B+C=3B+\frac{3B}{5}=\frac{18B}{5}=6$; whence $B=\frac{5}{3}$; $A=2B=\frac{10}{3}$; and $C=\frac{3B}{5}=1$: and therefore fluent of $\frac{6v^7\dot{v}}{(1+v^3)^3}$, $=$ fluent of $\frac{\frac{10}{3}v\dot{v}}{1+v^3}-\frac{\frac{5}{3}v^2}{1+v^3}-\frac{v^5}{(1+v^3)^2}$.

Again aſſume

$$D\times \text{h. l. }(1-v+v^2)-F\times \text{h. l. }(1+v)+E\times \text{flu. }\frac{\dot{v}}{1-v+v^5}=\text{flu. }\frac{\dot{v}v}{1+v^3};$$

taking the fluxions and dividing by $\dot{v}$, we get $\frac{2Dv+E-D}{1-v+v^2}-\frac{F}{1+v}=\frac{v}{1+v^3}$; multiplying by $1+v^3$, $v^2\times(2D-F)+v\times(D+E+F)+E-D-F=v$, or $v^2\times(2D-F)+v\times(D+E+F-1)+E-D-F=0$; and making the coefficients of the homologous terms $=0$, there will be had $2D=F$; $E=D+F=3D$; and $D+E+F=6D=1$; whence $D=\frac{1}{6}$, $E=3D=\frac{1}{2}$ $F=2D=\frac{1}{3}$, and therefore

$$\text{fluent of }\frac{v\dot{v}}{1+v^3}=\tfrac{1}{6}\times\text{h.l.}(1-v+v^2)-\tfrac{1}{3}\times\text{h.l.}(1+v)+\text{flu. }\frac{\dot{v}}{1-v+v^2}$$

$$=\tfrac{1}{3}\times\text{h. l. }\sqrt{\frac{1-v+v^2}{(1+v)^2}}+\text{flu. }\frac{\dot{v}}{1-v+v^2}$$

$$=\tfrac{1}{3}\times\text{h. l. }\sqrt{\frac{1+v^3}{(1+v)^3}}+\frac{R\sqrt{3}}{3}=\tfrac{1}{6}\times\text{h. l. }\frac{1+v^3}{(1+v)^3}+\frac{R\sqrt{3}}{3},$$

where R denotes the length of a circular arch radius 1, and tangent $\frac{v\sqrt{3}}{2-v}$; and, conſequently,

$$\text{fluent of }\frac{6v^7\dot{v}}{(1+v^3)^3}=\frac{10}{3}\times\text{flu. }\frac{\dot{v}v}{1+v^3}-\frac{\frac{5}{3}v^2}{1+v^3}-\frac{v^5}{(1+v^3)^2}$$

$$=\tfrac{5}{9}\times\text{h. l. }\frac{1+v^3}{(1+v)^3}+\frac{10R\sqrt{3}}{9}-\frac{\frac{5}{3}v^2}{1+v^3}-\frac{v^5}{(1+v^3)^2}$$

$= \cdot266866$ in the preſent caſe where $x=2$, or $v=x^{\frac{1}{6}}=2^{\frac{1}{6}}=1{\cdot}122462$.

Proceeding now to the ſecond expreſſion, put $\frac{x^3}{x^3-1}=v^3$, and taking the fluxions $\frac{-3x^2\dot{x}}{(x^3-1)^2}=3v^2\dot{v}$, or $\frac{x^2\dot{x}}{(x^3-1)^2}=-v^2\dot{v}$;

alſo

alſo $\frac{\dot{x}}{(x^3 - 1)^{\frac{1}{3}}} \pm \dot{v}$; multiplying theſe two expreſſions together,

$\frac{x^3\dot{x}}{(x^3 - 1)^{\frac{2}{3}}} = - v^3\dot{v}$; therefore fluent of $\frac{x^3\dot{x}}{(x^3 - 1)^{\frac{2}{3}}} = - \frac{v^4}{4}$.

$- \frac{v^4}{4} = 1{\cdot}5986011$, when $x = \frac{11}{10}$

$- \frac{v^4}{4} = 0{\cdot}2987187$, when $x = 2$,

therefore $1{\cdot}2998824$, or $1{\cdot}3$ nearly the difference of the preceding values of $- \frac{v^4}{4}$ is the fluent of $\frac{x^3\dot{x}}{(x^3 - 1)^{\frac{2}{3}}}$, generated whilſt x from $\frac{11}{10}$ increaſes to 2.

THIRD SOLUTION, *by Mr.* GRIFFITH WILLIAMS.

1. To find the fluent of $\frac{\dot{x}\sqrt[3]{x}}{(1 + \sqrt{x})^3}$.

Aſſume $x = v^6$, then $\sqrt[3]{x} = v^2$, $\sqrt{x} = v^3$, and $\frac{\dot{x}\sqrt[3]{x}}{(1 + \sqrt{x})^3} = \frac{6v^7\dot{v}}{(1 + v^3)^3}$ By Form. 7, Caſe 2, of Emerſon's Fluxions, the fluent of $\frac{v\dot{v}}{1 + v^3}$ is $= \frac{\sqrt{3}}{3} \times$ circ. arc. rad. 1 and ſine $\frac{\frac{1}{2}\sqrt{3}\cdot v}{\sqrt{(1 - v + v^2)}}$ $= \frac{1}{3} \times$ h. l. $\sqrt{(1 - v + v^2)} - \frac{1}{3} \times$ h. l. $(1 + v)$: put this fluent $=$ A; then by forms 12 and 14 of the ſame book, the fluent of $\frac{6v^7\dot{v}}{(1 + v^3)_3}$ is $= \frac{10}{3}$ A $- \frac{5}{3} \times \frac{v^2}{1 + v^3} - \frac{v^5}{(1 + v^3)^2}$, which expreſſion vaniſhes when $x = 0$, and when $x = 2$, the required fluent is $= {\cdot}266992$.

2. To find the fluent of $\frac{x^3\dot{x}}{(x^3 - 1)^{\frac{2}{3}}}$.

Aſſume $x^3 = \frac{v^3}{v^3 - 1}$; then $\frac{x^3\dot{x}}{(x^3 - 1)^{\frac{2}{3}}} = - v^3\dot{v}$, the fluent of which is $-\frac{v^4}{4}$; hence the correct fluent in this caſe is 1·3 nearly.

XXX. PRIZE QUESTION 90, *by* EDINBURGENSIS.

Suppoſe a horſe to move with a given velocity in the circumference of a given circle. Required the angle which the body of the horſe will make with the horizon by leaning towards the centre ?

FIRST SOLUTION, *by Mr.* THOMAS BAZLEY.

This queſtion appears to be equivalent to the following one. AGB (fig. 137, pl. 7,) is a given circle, and its centre is C; CE an axis perpendicular to the plane of the circle at C: AP is a right line loaded with a weight in the given point P: this right line is moveable upon the point A in a vertical plane paſſing through CE. Now when the right line AP is revolved in the circle AGB with a given velocity, it is required to determine the angle which AP makes with the plane of the circle, ſo that it may remain in equilibrio with reſpect to CB.

Draw the diameter ACB and EPF parallel thereto, drop the perpendicular PD on CA, and draw AF parallel to PD to meet EPF in F. Then it is plain from the compoſition of forces, that if PF is to PD, as the centrifugal force at P, to the force of gravity; then, PA has the required poſition.

This being premiſed, put $CA = r$, $AP = a$, $CD = x$, and $g = 16\frac{1}{12}$ feet, alſo $v =$ velocity of the point A. The velocity of A is to the velocity of P :: CA : CD or EP; hence, the velocity of $P = \frac{vx}{r}$. By the laws of central forces, we have, as the force of gravity is to the centrifugal force at P ſo is $2g : \frac{v^2x}{r^2}$ :: PD : PF or DA :: $\sqrt{\left\{a^2 - (r-x)^2\right\}} : r - x$; hence we obtain

$$4g^2(r-x)^2 = \frac{v^4}{r^4}x^2\left\{a^2 - (r-x)^2\right\}$$

and by reduction

$$x^4 - 2rx^3 + \left(\frac{4r^4}{v^4}g^2 - a^2 + r^2\right)x^2 - \frac{8r^5}{v^4}g^2x + 4g^2\frac{r^4}{v^4} = 0.$$

From this equation, x being determined, DP and DA become known; and, conſequently, the angle DAP which was required.

SECOND SOLUTION, *by* PHILOMATHES OXONIENSIS.

If the radius of the circle deſcribed by the centre of gravity of the horſe's body be denoted by a, the length of the arch of the circle deſcribed by the aforeſaid centre of gravity in one ſecond, by b, and the force of gravity by f; then, by mechanics *, the centrifugal force of the aforeſaid centre of gravity will be to the force of gravity as $\frac{bb}{a}$ to f. Now, if the ſine of the angle of the horſe's inclination to the horizon be called x, we ſhall have, by the reſolution of forces, and the condition of the queſtion, this equation, *viz.* $\frac{bbx}{a} = f\sqrt{1 - xx}$; and thence, $\frac{x}{\sqrt{(1 - xx)}} = \frac{af}{bb}$, the tangent of the angle required.

Example. If the radius of the circle be 15 feet, and the horſe's ſpeed at the rate of 5 miles per hour, b will be $= \frac{22}{3}$; and ſince f is $= 32\frac{1}{6}$, the tangent of the horſe's inclination to the horizon will be $= \frac{9 \times 15 \times 32\frac{1}{6}}{22 \times 22} = 8{\cdot}97211$, anſwering to 83° 38′$\frac{5}{12}$ nearly.

According to one or other of theſe methods were the Solutions given by Meſſrs. Codling, Collins, Jones, Myers, Pickering, Roville, Senex, *and* Wilton.

SENEX *is requeſted to ſend to Mr.* GLENDINNING'*s for the Medal for ſolving the Prize Queſtion.*

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* See the Ladies' Diary for 1783, p. 39.

NOTICES

RELATING TO MATHEMATICS.

I. DEATH OF DR. ROBISON.

IT is with regret that we announce the death of that juſtly celebrated mathematician and philoſopher, Dr. John Robiſon, Profeſſor of Natural Philoſophy in the Univerſity of Edinburgh. This event took place on the 30th of January laſt *. A few months before his death he gave to the world the firſt volume of a work, entitled, ELEMENTS OF MECHANICAL PHILOSOPHY, being the ſubſtance of a Courſe of Lectures which he had delivered annually in the diſcharge of his academical duty, ever ſince the year 1774. Such as have had the pleaſure of hearing theſe Lectures, or have read the numerous and valuable articles which he has contributed to the laſt edition of the Encyclopedia Britannica, will naturally ſuppoſe that this work muſt poſſeſs no common degree of excellence, and that it will far tranſcend moſt of the treatiſes that have appeared under the name of Elements of Natural Philoſophy for many years paſt in this country. We venture to aſſure them, that upon peruſing it they will not find their expectations diſappointed. A ſecond volume of the work we underſtand was in the preſs when the Author died; and it will be publiſhed in the courſe of the preſent ſummer.

Dr. Robiſon has been ſucceeded in the Profeſſorſhip of Natural Philoſophy by Mr. John Playfair, Profeſſor of Mathematics in the ſame Univerſity, with whoſe merits as a Mathematician and Philoſopher, as well as an elegant and correct writer, we cannot ſuppoſe any of our readers to be unacquainted.

Mr. Playfair has been ſucceeded in the Mathematical Chair by John Leſlie, Eſq. a gentleman of great Mathematical eminence, and Author of an *Experimental Inquiry into the Nature and Propagation of Heat*, a work which has juſtly acquired him great celebrity as an Experimental Philoſopher, and for which the Royal Society of London has awarded to him the biennial gold medal placed at their diſpoſal by the munificence of Count Rumford.

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* We ſuppoſe his age to have been about 65.

II. MATHEMATICAL WORKS IN THE PRESS.

IT has often been remarked as a ſingular circumſtance, that during the laſt half century, while the practice of mechanics and the ſtructure and operation of machines have received ſo many and ſuch valuable improvements in this country, we have only had one treatiſe (that by Emerſon) into which we can look for information both on the theory and the actual conſtruction of machinery. Mr. Gregory of the Royal Military Academy, Woolwich, has endeavoured to ſupply the deficiency juſt adverted to, and has now in the preſs a general Treatiſe of Mechanics, which is intended to be comprized in two volumes, octavo. The firſt volume will be devoted chiefly to the theory, and will be divided into five books, under the ſeveral heads of Statics, Dynamics, Hydroſtatics, Hydrodynamics, and Pneumatics. The ſecond volume will be chiefly appropriated to the deſcription of machinery, and will commence with ſome practical remarks on the application, improvement, and ſimplification of mechanical contrivances; on friction, the ſtiffness of ropes, the energy of different firſt movers, &c. And theſe will be followed by accounts, arranged alphabetically, of about one hundred of the moſt curious, uſeful, and important machines. In this latter part Mr. Gregory has been promiſed the aſſiſtance of ſome celebrated civil engineers; and the alphabetical arrangement (the only unfiniſhed part of the work,) will be completed in the courſe of the month of July, when he hopes he ſhall have viewed the communications of theſe gentlemen, or of any others who may favour him with deſcriptions of new and uſeful machines. The work is intended to be publiſhed before the end of the preſent year.

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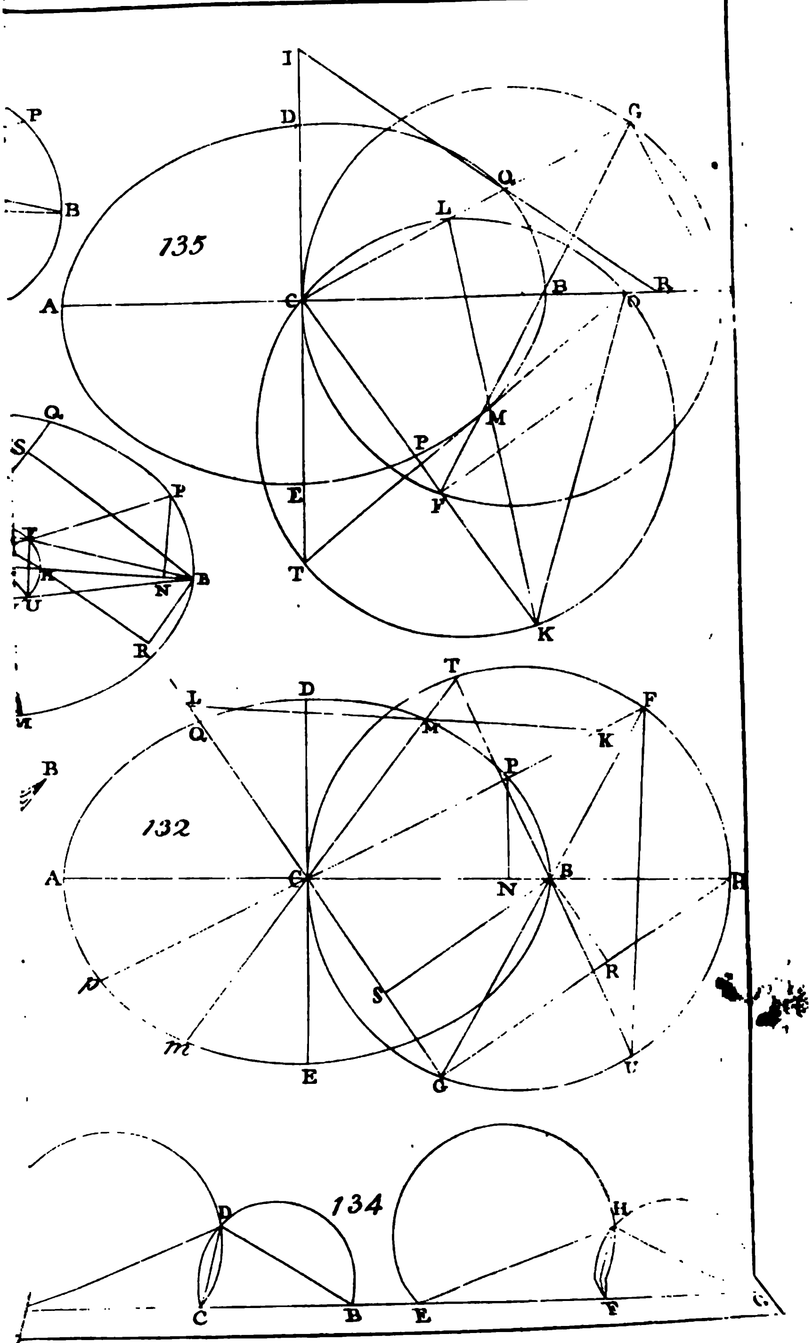
Plate VII. *Fig* 126 *to* 137.



ARTICLE IV.

Solutions to Queſtions propoſed in Number III.

I. QUESTION 91, *by Capt.* GEO. GORRY.

Paſſing along a ſtraight and level road, near a very lofty tower, and on the ſame horizontal plane with its baſe, I wiſhed to aſcertain its height; but being prevented by fences from quitting the road, and having no inſtrument but one for taking vertical angles, I proceeded thus: at a convenient point on the road, I found the angle of elevation of the top of the tower, 30° 48′; then meaſuring 60 yards along the road, I found a ſecond angle of elevation 40° 33′; at the end of another 60 yards in the ſame direction, I intended taking a third angle of elevation, but the tower was hidden by a high wall; I therefore meaſured on to 72 yards from the ſecond ſtation, and there found the angle of elevation of the top of the tower, 50° 23′. From theſe data, the height of the tower, and its neareſt diſtance from each of the three ſtations are required?

FIRST SOLUTION, *by Mr.* WM. MELLOR, *London.*

Suppoſe AED (fig. 164. pl. 9.) the horizontal plane, EF the tower perpendicular thereto; let A, B, C be the three places of obſervation. Join AE, BE, CE; AF, BF, CF; and on AC demit the perpendicular ED. Then we have given $AB = 60$ and $BC = 72$ yards; the angle $FCE = 50°\ 23'$, the angle $FBE = 40°\ 33'$, and the angle $FAE = 30°\ 48'$, to find the diſtances AE, BE, CE; and EF the height of the tower.

Since the triangles AEF, BEF, CEF are all right angled at E, it is obvious that the diſtances AE, BE, CE will be as the co-tangents of the given angles FAE, FBE, FCE: let theſe co-tangents be repreſented by a, b, c reſpectively; then we have to determine the point E ſuch that the lines AE, BE, CE may have the ſame ratio as a, b, c.

This may be effected a variety of ways, but as ſimple a method as any is that in Simpſon's Select Exerciſes, page 130; which is nearly as follows.

Take BM equal to BC, BN to AB, and make

$MG : BM\,(BC) :: a : b$, and

$BN\,(AB) : NG :: b : c$.

With the lines MN, MG, NG form the triangle MNG, and join BG. Draw AE to make the angle EAB = MGB, meeting BG produced in E the point required.

For by the ſimilar triangles AEB, GMB,

AE : BE :: MG : MB :: a : b; and
BE : BA (BN) :: MB : BG.

Therefore the triangles BEC, BGN are similar, and therefore BE : EC :: BN : NG :: b : c; wherefore it is manifeſt that the diſtances AE, BE, CE are reſpectively as a, b, c.

The calculation follows very eaſily from the conſtruction; for in the triangle MGN all the ſides are given to find the angle GMN, which is equal to the angle AEB; then in the triangle MGB two ſides and the included angle are given to find the angle MGB, which is equal to the angle EAB. Hence in the triangle AEB we find AE = 159·087, BE = 110·8414; and then CE = 78·507 yards. Finally, in the right-angled triangle AEF we have the side AE and the angle FAE, to find EF, the height of the tower, equal to 94·835 yards.

SECOND SOLUTION, *by Mr.* BAZLEY, *Bolton.*

Let AB = 60 = r, BC = 72 = s; alſo let the cotangents of the angles FAE (30° 48′), FBE (40° 33′), FCE (50° 23′), be repreſented by the letters a, b, c reſpectively.

Then, by trigonometry, putting EF = x, we have,
$1 : a :: x : ax$ = AE, $1 : b :: x : bx$ = BE, $1 : c :: x : cx$ = CE; and, Euc. 12. 11. $a^2x^2 = b^2x^2 + r^2 + 2r \times \text{BD}$; hence

$$\text{BD} = \frac{a^2x^2 - b^2x^2 - r^2}{2r}.$$ In like manner

$$\text{CD} = \frac{b^2x^2 - c^2x^2 - s^2}{2s} = \text{BD} - s: \text{ whence BD} = \frac{b^2x^2 - c^2x^2 + s^2}{2s}.$$

Therefore $$\frac{b^2x^2 - c^2x^2 + s^2}{2s} = \frac{a^2x^2 - b^2x^2 - r^2}{2r}.$$ Hence

$$x^2 = \frac{rs^2 + sr^2}{s(a^2 - b^2) - r(b^2 - c^2)}, \text{ and } x = \sqrt{\frac{rs\,(r+s)}{s(a^2 - b^2) - r(b^2 - c^2)}} = 94·835$$

yards. Alſo AE = 159·087, BE = 110·8414, and CE = 78·507 yards.

The Propoſer alſo anſwered it.

II. QUESTION 92, *by* NEGOTIATOR.

A ſells goods to B, on condition to allow 10 per cent. diſcount, if he paid in ſix months.—What diſcount ought A to allow, if payment be made him in two months; intereſt of money being 5 per cent. per annum, ſimple intereſt?

FIRST SOLUTION, *by Mr.* BAZLEY.

If s be put to denote the value of the goods, hen the diſcounted ſum paid at the end of ſix months is $\frac{9}{10}\,s$. Let x be the ſum which B ought to pay if the goods be paid for at the end of two months; then it is plain that x at ſimple interest for the remaining four months ſhould amount to $\frac{9}{10}\,s$: that is (by the rate of intereſt given by the queſtion,

$$x + \frac{x}{60} = \frac{9}{10}\,s.$$

From this equation we have $x = \frac{54s}{61}$; and, conſequently, $s - x = \frac{7s}{61}$, or £11·4754 per cent. the diſcount required.

SECOND SOLUTION, *by* A. M.

Since B is to pay only 90 per cent. for his goods if he pays A in ſix months, therefore the ſum per cent. which B ought to pay if payment be made to A in two months, is ſuch a ſum as being put out at 5 per cent. ſimple intereſt for four months will amount to 90 £. Hence $90 \div (\cdot\dot{3} \times \cdot 05) + 1 = 88\cdot524$ £. = 88 £. 10s. 5¾ *d.* per cent. B ought to give A for his goods if payment be made in two months, or 100 £.—88 £. 10s. 3¾*d.* = 11 £. 9*s.* 6¼ *d.* per cent. is the diſcount required.

Mr. Mellor *anſwered this queſtion.*

III. QUESTION 93, *by Mr.* WM. SIMPSON, *Bolton.*

If a circle be deſcribed through the two foci of an ellipſe, and any point in the conjugate axis produced; then a right line being drawn from that point, to one of the points where the circle cuts the ellipſe, will be a tangent at that point. Required the demonſtration?

SOLUTION, *by Mr.* J. JOHNSON, *Birmingham.*

Fig. 165. pl. 9. Let a circle which paſſes through the foci f, F, meet the conjugate axis in E and H and the ellipſis in P, and draw fP, FP, HP, and let EP produced meet the tranſverſe axis in G. Then it is evident that the arc fHF is biſected in H, therefore the angle FPH is equal to the angle SPH ; and the angles HPG, HPE are, each of them, right angles. If the former angles be taken from the latter, the remaining angle FPG will be equal to the remaining angle fPE, therefore EP is a tangent to the ellipſe as is proved by all the writers on conic ſections.

Several properties of the ellipſe may be immediately deduced from the known properties of the triangle fPF and its circumſcribing circle FPEf. We ſelect a few of the moſt obvious.

1. $fP : FP :: fK : FK$.
2. $fP \cdot PF = KP \cdot PH = IC \cdot HE$.
3. Let HL be drawn perpendicular to fP. Then LP = the ſemi-tranſverſe axis = half the ſum of fP, PF ; and FL = half the difference of fP, PF.
4. If LC be drawn to the centre, it will be parallel to the tangent at P.
5. Let PI be parallel and PO perpendicular to AB.
   Then $fP + PF$ or $AB : fF :: 2CO : fP - PF$;
   or $AB : fF :: CO : fL$.
6. $AC^2 = LP^2 = HI \cdot EC$.
7. $(fP - PF)^2 = CH \cdot IE$.
8. $AC^2 : CD^2 :: HP : PK$.

Meſſrs. Bazley, Mellor, *and* A. M. *likewiſe ſent ſolutions to this queſtion.*

IV. QUESTION 94, *by* CENTROBARICOS.

Suppoſe a cone, whoſe axis is 10, and diameter 8 inches, be made to vibrate, its vertex being the point of ſuſpenſion; it is required to aſcertain at what diſtance from its vertex, *on the ſlant ſide,* it muſt ſtrike an immoveable obſtacle, ſo that all its angular motion ſhall be deſtroyed, or that its point of ſuſpenſion ſhall not be affected by the ſtroke ?

SOLUTION, *by Mr.* BAZLEY.

Let ABC (fig. 166. pl. 9.) be the cone whoſe vertex is C, and axis CD: then if O be the centre of percuſſion and OP perpendicular to the ſide CB, it is plain from the principles of mechanics (Emerſon's Mechanics 4, to prop. 9.) that P is the required point.

Now it is ſhewn by the writers on fluxions that the diſtance $\mathrm{CO} = \frac{4\mathrm{CD}^2 + \mathrm{DB}^2}{5\mathrm{CD}}$, and by the ſimilar triangles COP, CDB, we have

$$\mathrm{CB} : \mathrm{CD} :: \mathrm{CO} : \mathrm{CP} :: \frac{4\mathrm{CD}^2 + \mathrm{DB}^2}{5\mathrm{CD}} : \frac{4\mathrm{CD}^2 + \mathrm{DB}^2}{5\mathrm{CB}}.$$

In the preſent caſe $\mathrm{CD} = 10$, $\mathrm{DB} = 4$, $\mathrm{CB} = \sqrt{(100 + 16)} = 10{\cdot}77033$ and $\mathrm{CP} = \frac{4\mathrm{CD}^2 + \mathrm{DB}^2}{5\mathrm{CB}} = 7{\cdot}725$ inches, the diſtance required.

V. QUESTION 95, *by* DIOPHANTUS.

It is required to find three rational numbers, x, y, and z, ſuch that the formula

$$x^2 + y^2 + 2z^2, \; x^2 + z^2 + 2y^2, \text{ and } y^2 + z^2 + 2x^2,$$

may all be ſquares ?

SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

Put $z + m + n = x$, and $z + m = y$, then

$x^2 + y^2 + 2z^2 = 4z^2 + 2z(2m + n) + 2m^2 + 2mn + n^2 =$ a ſquare, and

$x^2 + z^2 + 2y^2 = 4z^2 + 2z(3m + n) + 3m^2 + 2mn + n^2 =$ a ſquare.

Aſſume $2z + r$ for the root of the former, and $2z + s$ for the m root of the latter of these expreſſions, that is, put

$4z^2 + 2z(2m + n) + 2m^2 + 2mn + n^2 = (2z + r)^2 = 4z^2 + 4rz + r^2$, and

$4z^2 + 2z(3m + n) + 3m^2 + 2mn + n^2 = (2z + s)^2 = 4z^2 + 4sz + s^2$,

from whence

$$z = \frac{2m^2 + 2mn + n^2 - r^2}{2(2r - 2m - n)} = \frac{3m^2 + 2mn + n^2 - s^2}{2(2s - 3m - n)}$$

equating the numerators and denominators of theſe fractions to make them identical, we ſhall thence obtain

$$s^2 - r^2 = m^2, \text{ and } s - r = \frac{m}{2}$$

dividing the former of theſe by the latter, $s + r = 2m$; whence $s = \frac{5m}{4}$; and $r = \frac{3m}{4}$, and by means of theſe

$$z = \frac{2m^2 + 2mn + n^2 - r^2}{2(2r - 2m - n)} = \frac{2m^2 + 2mn + n^2 - \frac{9m^2}{16}}{-(m + 2n)} = \frac{23m^2 + 32mn + 16n}{-16(m + 2n)},$$

$$y = z + m = \frac{23m^2 + 32mn + 16n^2}{-16(m + 2n)} + m = \frac{16m^2 + 16n^2}{-16(m + 2n)},\ \text{and}$$

$$x = z + m + n = \frac{7m^2 - 16mn - 16n^2}{-16(m + 2n)}.$$

Omitting the common denominator $-16(m + 2n)$ we may take

$$z = 23m^2 + 32mn + 16n^2,$$
$$y = 7m^2 + 16n^2 \text{ and}$$
$$x = 7m^2 - 16mn - 16n^2;$$

theſe values make the two firſt of the propoſed expreſſions ſquares. Therefore there only remains the laſt of the ſaid expreſſions, *viz.* $y^2 + z^2 + 2x^2$, to be made a ſquare.

By writing for x, y and z the values juſt deduced, we ſhall have, $y^2 + z^2 + 2x^2$

$$= (7m^2 + 16n^2)^2 + (23m^2 + 32mn + 16n^2)^2 + 2(7m^2 - 16mn - 16n^2)^2$$
$$= 676m^4 + 1024m^3n + 2048m^2n^2 + 2048mn^3 + 1024n^4$$
$$= 4 \times (169m^4 + 256m^3n + 512m^2n^2 + 512mn^3 + 256n^4)$$

= a ſquare. Therefore

$169m^4 + 256m^3n + 512m^2n^2 + 512mn^3 + 256n^4$ = a ſquare.

Aſſume $13m^2 + \frac{128}{13}mn + 16n^2$ for its root; that is, put

$$169m^4 + 256m^3n + 512m^2n^2 + 512mn^3 + 256n^4$$
$$= \left(13m^2 + \frac{128}{13}mn + 16n^2\right)^2$$
$$= 169m^4 + 256m^3n + \frac{(128)^2m^2n^2}{169} + 416m^2n^2 + \frac{4096mn^3}{13} + 256n^4$$

which after proper reduction gives $\frac{m}{n} = \frac{208}{1}$, take $m = 208$, then $n = 1$, and hence

$$x = 7m^2 - 16mn - 16n^2 = 16 \times 18719$$
$$y = 7m^2 + 16n^2 = 16 \times 18929$$
$$z = 23m^2 + 32mn + 16n^2 = 16 \times 62609.$$

And as theſe numbers are all diviſible by 16, we may take $x = 18719, y = 18929$ and $z = 62609$.

VI. QUESTION 96, *by Mr.* JOHN DAWES, *Birmingham.*

In Mr. Watt's ſteam engine, the reciprocating motion of the engine, is converted into a circular motion by a peculiar contrivance, conſiſting of two equal wheels that work into one another: it is required to demonſtrate that the central wheel will turn round twice for every revolution of the planet wheel about it, or for every complete ſtroke of the engine; and alſo to define what will be the motion of the central wheel when it is not equal to the planet wheel, but in any given proportion to it?

FIRST SOLUTION, *by Mr.* DAWES, *the Propoſer.*

The ratio of the times of revolution of the central and planet-wheels, is equal to the ratio of the radius of the central wheel, and the ſum of the radii of the central and planet wheels.

Let C, (fig. 167. pl. 9.) repreſent the central, and P, L, A, N, four poſitions of the planet-wheel; then, it is evident that the centre of the planet-wheel revolves in a circle whoſe radius is equal to the ſum of the radii of the central and planet-wheels; and (Emerſon's Algebra, page 450) the locus of any point (p) in the circumference of the planet wheel (which has no motion round its own axis) is a circle equal to that which is the locus of the centre of the ſaid wheel; but as both wheels, at the point of contact, move with the ſame velocity, it follows that any two points, one in the periphery of each wheel, muſt move through the ſame ſpace in equal times; therefore, in equal times, it will be inverſely, as the revolutions of the central, is to the circumvolutions of the planet, ſo is the radius of the central, to the radius of the circle formed by either the centre or any point in the extremity of the planet-wheel. *Q. E. D.*

Obſervation. The above demonſtration is upon the ſuppoſition that the connecting rod which ſuſpends the planet-wheel, always moves parallel to itſelf; but in reallity the end is connected to the beam, which moves upon a fixt fulcrum, therefore it cannot move parallel to itſelf, but makes an angle at the end of the beam which deſtroys much of the power of the engine.

SECOND SOLUTION, *by Mr.* BAZLEY.

Let CAF (fig. 168, pl. 9.) be the central-wheel, C the centre, AOB, a diameter of the revolving or planet-wheel, O, its centre which deſcribes the circle OE, concentric to AF. Suppoſe the wheels faſtened together at their point of contact A, and let the motion commence at the poſition CAB. Now if the whole thus fixed be conceived to deſcribe round C the angle BCD, then it is plain that the diameter AB will have come to the poſition FD. But by the nature of the queſtion the diameter AB is moved always parallel to itſelf; and therefore that FD may have the poſition *ba*, parallel to BA, it is evident that the arc *a*F of the planet-wheel muſt impel the central-wheel through an equal arc; ſo that the arc through which the central-wheel has paſſed, whilſt the centre of the planet-wheel has deſcribed the arc OE, is AF + F*a*: and it remains to find the angle anſwering to that arc of the central-wheel. Put $2\text{AC} = a$, $\text{AB} = b$, and the $\angle \text{OEC} = \text{FE}a = \text{ACF} = n$; then, by the nature of the circle, $a : b :: n : \frac{bn}{a}$ = the angle of the central whoſe arc is $= \text{F}a$; hence, whilſt the revolving wheel has paſſed through the $\angle n$, the central-wheel has deſcribed the $\angle \left(n + \frac{bn}{a}\right) = \frac{a+b}{a} \times n$; ſo that whilſt the planet wheel makes one revolution, the central-wheel makes $\frac{a+b}{a}$ turns on its axis. If a and b, are equal, $\frac{a+b}{a} = 2$.

VII. QUESTION 97, *from Geometrie Deſcriptive, by* MONGE.

If four unequal ſpheres be any how ſituated in ſpace, and every two of them be circumſcribed by a cone, ſo that its vertex may be on the ſame ſide of the centres of the ſpheres; the vertices of the ſix cones thus conſtituted ſhall be at the interſections of four ſtraight lines ſituated in a plane: and if every two of the ſpheres be circumſcribed by another cone, having its vertex between the centres of the ſpheres; the vertices of theſe cones, taken three by three, ſhall be in the ſame plane as three of the former. Required the demonſtration?

SOLUTION,

SOLUTION, *by Mr.* LOWRY, *R. M. College.*

Let A, B, C, D, (fig. 169, pl. 9.) be the centres of four unequal ſpheres any how ſituated in ſpace; P, Q, R, S, T, U, the vertices of the cones circumſcribing every two of them taken in the order A, B; A, C; A, D, &c. each vertex being on the ſame ſide of the centres of the ſpheres; and *p*, *q*, *r*, *s*, *t*, *u*, the vertices of the circumſcribing cones, taken in the ſame order, but ſituated between the centres of the ſpheres.

Then, it is evident that each of theſe vertices is in the ſtraight line joining the centres of the two ſpheres which are circumſcribed by the reſpective cones. Let the lines PBA, RDA, TDB, be drawn from the vertices P, R, T, to the centres of the ſpheres A, B, D, and conceive a plane to paſs through the points, A, B, D; then it is evident that the vertices, P, R, T, will alſo be in that plane. Join PR, TR, and draw BV parallel to AD meeting PR in V, and let the radii of the ſphere whoſe centres are A, B, C, D, be denoted by *a*, *b*, *c*, *d*, reſpectively.

Then by the property of tangents,

$a : b :: AP : BP :: AR : BV$, and
$a : d :: AR : DR$; therefore
$b : d :: BV : DR$; but
$b : d :: BT : DT$; therefore
$BV : DR :: BT : DT$. From whence it is evident that the points P, V, R, T, are in a ſtraight line.

Hence, it follows, that if three unequal ſpheres be any how ſituated in ſpace, and every two of them be circumſcribed by a cone, the vertices of the three cones thus conſtituted will be in the ſame ſtraight line.

From this we may infer that the vertices P, Q, S, of the cones circumſcribing the ſpheres A, B, C, are in the ſtraight line PS; that the vertices U, Q, R, of the cones circumſcribing the ſpheres A, C, D, are in the ſtraight line UR; and that the vertices U, S, T, of the cones circumſcribing the ſpheres B, C, D, are in the ſtraight line UT. Therefore the vertices of the ſix circumſcribing cones are at the interſections of the four ſtraight lines PT, PS, UR, and UT.

And theſe ſtraight lines are all in the ſame plane: for if a plane be conceived to paſs through the points P, T, U, the points R, S, will neceſſarily be in that plane, and conſequently the point Q will alſo be in that plane.

Again, if BV meet P*r* in *v*, it will appear, exactly as in the firſt part of the demonſtration, that the vertices P, *t*, *r*, are in the ſtraight line P*r*, and that the vertices T, *r*, *p*, are in the ſtraight line Tp; therefore it is obvious that the vertices p, t, r, are in the ſame plane as the vertices P, T, R, or in the plane paſſing through the centres A, B, D. In the ſame manner it appears that the vertices p, q, s, are in the ſame plane as the vertices P, Q, S; that the vertices u, q, s, are in the ſame plane as the vertices U, Q, S; and, finally, that the vertices u, s, t, are in the ſame plane as the vertices U, S, T.

Corol. It follows immediately from the above, that "If the tangents drawn to every two of three unequal circles be produced till they meet, the points of interſection will be a in right line."

The next Ten Queſtions, taken from the Cambridge University Calendar, are ſelected from the Senate Houſe Problems, propoſed to Candidates for the Degree of B. A. *in January,* 1804.

VIII. QUESTION 98, *from the* CAMB. UN. CAL.

Three known rectilinear objects, placed contiguous, and in the ſame right line, appeared to the eye of a ſpectator of the ſame length; find his poſition?

First Solution, *by Mr.* Mellor, *London.*

Let AB, BC, CD, (fig. 170, pl. 9.) repreſent the three rectilinear objects, placed contiguous and in the ſame ſtraight line. And let E be the poſition of the ſpectator's eye when the objects appear of the ſame length. Draw AE, BE, CE, DE. Then by the principles of optics, the objects AB, BC, CD, will be apparently of the ſame length when the ſubtended angles AEB, BEC, CED, are equal.

Therefore, Euc. vi. 3, AE : CE :: AB : BC
and BE : DE :: BC : CD:

Produce AC to G, ſo that CG : AG :: BC : AB, and on BG as a diameter deſcribe a circle B, E, G, E′, which is well known to be the locus of E, when AE has to CE the ſame ratio as AB to BC. Again produce BD to H, ſo that DH : BH :: BC : CD, and a circle deſcribed on the diameter CH will be the locus of the point E when BE has to ED the ſame ratio as BC to CD; therefore the interſection of theſe circles will determine the point E.

If a circle be deſcribed on the line E E′, which joins the points of interſection of the circles, as a diameter, and having its plane perpendicular to the line AH, it is obvious that at any point whatever in the circumference of this circle the three objects will appear of the ſame length.

SECOND SOLUTION, *by Mr.* BAZLEY.

Put AB $= a$, BC $= b$, CD $= c$, AE $= v$, BE $= x$, CE $= y$, and DE $= z$.

Then $a : b :: v : y = \frac{bv}{a}$, and $b : c :: x : z = \frac{cx}{b}$;
Alſo (by Playfair's Euclid, prop. B, book vi.)
$vy = ab + x^2 = \frac{bv^2}{a}$; hence $v^2 = \frac{a^2b + ax^2}{b}$. Again
$zx = bc + y^2$, or $\frac{cx^2}{b} = bc + \frac{b^2v^2}{a^2} = bc + \frac{ab^2 + bx^2}{a}$; or
$cax^2 = ab^2c + ab^3 + b^2x^2$: whence $x = \sqrt{\frac{ab^2c + ab^3}{ca - b^2}}$;
from which it appears that if b^2 exceeds ca, that is, if BC^2 be greater than AB × CD, the thing is impoſſible.

IX. QUESTION 99, *from the* CAMB. UN. CAL.

Shew how the circumference of an ellipſe may be computed, of which the excentricity is ·99, the ſemi-axis major being 1?

SOLUTION.

In conſequence of the ſerieſes commonly given, in our Treatiſes of Fluxions, &c. for finding the perimeter of an ellipſe, converging ſo very ſlowly when the excentricity is great as to be entirely uſeleſs, Mathematicians have been induced to conſider the ſubject at greater length, which has led to the diſcovery of ſeveral ſeries that converge quickly for every degree of excentricity that can poſſible take place. As ſome of theſe ſeries may probably be unknown to many of the readers of the Repoſitory, it may not be improper to ſelect ſuch of them as are moſt remarkable for their elegance or practical utility.

First Series, by Mr. Ivory, inſerted in the 4th. Vol. of the Tranſactions of the Royal Society of Edinburgh.

Half the perimeter =

$$\frac{\pi}{1+d}\left\{1+\frac{1^2}{2^2}\cdot d^2+\frac{1^2\cdot 1^2}{2^2\cdot 4^2}\cdot d^4+\frac{1^2\cdot 1^2\cdot 3^2}{2^2\cdot 4^2\cdot 6^2}\cdot d^6+\&c.\right\}*$$

where 1 and c are the ſemi-axes, $d=\frac{1-c}{1+c}$, and $\pi = 3{\cdot}14159$.

This ſeries converges with great rapidity when the excentricity is ſmall.

Second Series, given by Legendre in *Mem de l'Acad.* and by Euler, *Novi Comm. Petrop*, Tom. XVIII.

One fourth of the perimeter =

$$\sqrt{\left(1-\frac{e^2}{2}\right)}\frac{\pi}{2}\left\{1-\frac{1\cdot 1}{2\cdot 4}\cdot\tfrac{1}{2}c^2-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{1\cdot 3}{2\cdot 4}c^4-\&c.\right\}$$

where e is the excentricity, and $c=\frac{e^2}{2-e^2}$.

This ſeries is not very commodious for arithmetical computation when e is nearly equal to 1.

Third Series, given by Euler in his *Animadverſiones in Rectificationum Ellipſeos*, which forms a part of the ſecond volume of his Opuſcula.

One fourth of the perimeter is =

$$1+(c^2+\frac{3}{4}\cdot\tfrac{1}{2}c^4+\frac{3\cdot 5}{4\cdot 6}\cdot\frac{1\cdot 3}{2\cdot 4}c^6+\frac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}c^8+\&c.)$$

$$\text{hyp. log. }\frac{2}{\sqrt{c}}$$

$$-\ c^2\cdot\tfrac{1}{4}$$

$$-\ \frac{3}{4}\cdot\tfrac{1}{2}c^4\left(\frac{7}{12}-\frac{1}{2\cdot 3\cdot 4}\right)$$

$$-\ \frac{3\cdot 5}{4\cdot 6}\cdot\frac{1\cdot 3}{2\cdot 4}c^6\left(\frac{13}{20}-\frac{1}{2\cdot 3\cdot 4}-\frac{1}{4\cdot 5\cdot 6}\right)$$

$$-\ \frac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}c^8\left(\frac{19}{28}-\frac{1}{2\cdot 3\cdot 4}-\frac{4}{4\cdot 5\cdot 6}-\frac{1}{6\cdot 7\cdot 8}\right)$$

$$-\ \&c.$$

where 1 and c are the ſemi-tranſverſe and ſemi-conjugate axes.

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* *Mr. Ivory's* Paper, which contains the inveſtigation of this ſeries is *re-publiſhed in the firſt* part of the Mathematical Tracts and Selections.

This is a very uſeful ſeries, and converges ſo much the faſter as the excentricity is more nearly equal to 1.

Fourth Series, given by Mr. Wallace, in the 5th volume of the Edinburgh Tranſactions.

Half the perimeter is $= \pi P (1 - eQ)$.

Where $e =$ the excentricity, $\pi = 3{\cdot}14159$

$$e' = \frac{1 - \sqrt{1 - e^2}}{1 + \sqrt{1 - e^2}},\ e'' = \frac{1 - \sqrt{1 - e'^2}}{1 + \sqrt{1 - e'^2}},\ e''' = \frac{1 - \sqrt{1 - e'^2}}{1 + \sqrt{1 - e'^2}},\ \&c.$$

$$P = (1 + e')(1 + e'')(1 + e''')(1 + e^{\text{iv}}),\ \&c.$$

$$\text{and } Q = \frac{e}{2} + \frac{e\,e'}{2.2} + \frac{e\,e'\,e''}{2.2.2} + \frac{e\,e'\,e''\,e'''}{2.2.2.2} + \&c.$$

Or,

if α be an arc ſuch that ſin. $\alpha = e$,

then $e' = \text{tang.}^2 \frac{\alpha}{2}$, and $1 + e' = \text{sec.}^2 \frac{\alpha}{2}$.

Again, by taking α' ſuch that ſin. $\alpha' = e'$,

we have $e'' = \text{tang.}^2 \frac{\alpha'}{2}$, and $1 + e'' = \text{ſec.}^2 \frac{\alpha'}{2}$; and by continuing in this manner as far as may be neceſſary we have

$$P = \text{ſec.}^2 \frac{\alpha}{2} \,.\, \text{ſec.}^2 \frac{\alpha'}{2} \,.\, \text{ſec.}^2 \frac{\alpha''}{2},\ \&c.$$

$$Q = \frac{\text{ſin.}\,\alpha}{2} + \frac{\text{ſin.}\,\alpha \,.\, \text{ſin.}\,\alpha'}{2\,.\,2} + \frac{\text{ſin.}\,\alpha \,.\, \text{ſin.}\,\alpha' \,.\, \text{ſin.}\,\alpha''}{2\,.\,2\,.\,2} + \&c.$$

Where the values of P and Q may be computed with great facility by means of the common trigonometrical tables.

This ſeries has the advantage of converging quickly in every caſe of excentricity that can occur, but more eſpecially when it does not exceed $\sqrt{\tfrac{1}{2}}$.

Fifth Series, by Mr. Woodhouse, inſerted in the Tranſactions of the Royal Society of London, for the year 1804.

One fourth of the perimeter is =

$$1 - b\left\{\frac{b.(1+{}'b)}{2\,.\,2} + \frac{b.{}'b.(1+{}'b).(1+{}''b)}{2\,.\,2\,.\,2\,.\,2} + \&c.\right\} + \frac{b}{2m}.{}'P.{}'Q.\text{h.l.}\frac{4}{(m)b}$$

Where 1 and b are the ſemi-axes,

$${}'b = \frac{b^2}{\left\{1 + \sqrt{(1 - b^2)}\right\}^2},\ {}''b = \frac{{}'b^2}{\left\{1 + \sqrt{(1 - {}'b^2)}\right\}^2}\ \&c.$$

$${}'P = (1 + {}'b)(1 + {}''b)(1 + {}'''b)\ldots\ldots(1 + \beta).$$

$${}'Q = \frac{b}{2} + \frac{b}{2} \,.\, \frac{{}'b}{2} + \frac{b}{2} \,.\, \frac{{}'b}{2} \,.\, \frac{{}''b}{2} + \&c.$$

and ${}^{(m)}b$, the term of the ſeries $'b$, $''b$, $'''b$, &c. after which, without ſenſible error, each ſucceeding term may be taken equal to the ſquare of the preceding term. This formula, like the preceding, is generally applicable, and particularly when the excentricity exceeds $\sqrt{\tfrac{1}{2}}$.

For the inveſtigation of theſe ſeries, we muſt refer the reader to the reſpective works already cited, or to the learned memoir of the laſt-mentioned author, where the ſeveral inveſtigations are given, and their mutual relation to each other pointed out.

We ſhall now exhibit the computation of the periphery of an ellipſe, whoſe excentricity is ·99 and ſemi-axis major 1, as required by the queſtion.

And firſt by the third Series,

$$c^2 = 1 - (\cdot 99)^2 = \cdot 0199;\quad \frac{3}{8}c^4 = \cdot 0001485;\quad \frac{15}{64}c^6 = \cdot 0000019;$$

$$\tfrac{1}{4}c^2 = \cdot 0049750;\quad \frac{13}{24}\cdot\frac{3}{8}c^4 = \cdot 0000804;\quad \frac{3}{5}\cdot\frac{15}{64}c^6 = \cdot 0000012;$$

and hyp. log. of $\frac{2}{\sqrt{c}} = 1\cdot 6724061$. Hence

$$1 + \left(c^2 + \frac{3}{8}c^4 + \frac{15}{64}c^6\right) \times \text{h. l.}\ \frac{2}{\sqrt{c}} = +\ 1\cdot 0335324$$

$$-\tfrac{1}{4}c^2 - \frac{13}{24}\cdot\frac{3}{8}c^4 - \frac{3}{5}\cdot\frac{15}{64}c^6 = -\ 0\cdot 0050566$$

one-fourth of the periphery $= 1\cdot 0284758$

Multiply by 4

And the perimeter of the ellipſe is $= 4\cdot 1139032$.

Secondly, by the fourth ſeries.

1. Calculation of the logarithms of e', $1 + e'$, $1 + e''$, &c.

Logarithms.

$e = \cdot 99 = \text{ſin.}\ \alpha = \text{ſin.}\ 81^\circ\ 53'\ 25''\cdot 33$ $\quad \bar{1}\cdot 9956352$

$e' = \tan.^2 \frac{\alpha}{2} = \text{ſin.}\ \alpha' = \text{ſin.}\ 48^\circ\ 49'\ 43''\cdot 22$ $\quad \bar{1}\cdot 8766476$

$1 + e' = \sec.^2 \frac{\alpha}{2}$ $\quad 0\cdot 2437186$

$e'' = \tan.^2 \frac{\alpha'}{2} = \text{ſin.}\ \alpha'' = \text{ſin.}\ 11^\circ\ 53'\ 26''\cdot 44$ $\quad \bar{1}\cdot 3139620$

$1 + e' = \sec.^2 \frac{\alpha'}{2}$ $\quad 0\cdot 0813636$

$e''' =$

		Logarithms.
$e''' = \tan.^2 \frac{\alpha''}{2} = \text{ſin. } \alpha''' = \text{ſin. } 0^\circ\ 37'\ 17''{\cdot}00$		$\bar{2}{\cdot}0352326$
$1 + e''' = \text{ſec.}^2 \frac{\alpha''}{2}$		$0{\cdot}0046846$
$e^{\text{IV}} = \tan.^2 \frac{\alpha'''}{2} = \text{ſin. } \alpha^{\text{IV}} =$		$\bar{5}{\cdot}4684310$
$1 + e^{\text{IV}} = \text{ſec.}^2 \frac{\alpha'''}{2}$		$0{\cdot}0000128$

1. Computation of $\pi P (1 - eQ)$

$\frac{e^2}{2}$	$\cdot4900500$	$\bar{1}{\cdot}6902404$
$\frac{e^2 e'}{2 . 2}$	$\cdot1844412$	$\bar{1}{\cdot}2658580$
$\frac{e^2 e' e''}{2 . 2 . 2}$	$\cdot0190016$	$\bar{2}{\cdot}2787900$
$\frac{e^2 e' e'' e'''}{2 . 2 . 2 . 2}$	$\cdot0001030$	$\bar{4}{\cdot}0129926$
	$\cdot6935958$	
	$1\cdot$	
$(1 - e Q) =$	$\cdot3064042$	$\bar{1}{\cdot}4862947$
$1 + e'$ - - - - -		$0{\cdot}2437186$
$1 + e''$ - - - - -		$0{\cdot}0813696$
$1 + e'''$ - - - - -		$0{\cdot}0046846$
$1 + e^{\text{IV}}$ - - - - -		$0{\cdot}0000128$
π - - - - -		$0{\cdot}4971499$
$\pi P (1 - e Q) =$	$2{\cdot}056952$	$0{\cdot}3132242$
	2	
Perimeter $=$	$4{\cdot}113904$	

Thirdly, by the fifth ſeries.

$$b = \surd(1 - (\cdot99)^2) = \cdot1410673$$

$$'b = \frac{b^2}{\left\{1 + \surd(1 - b^2)\right\}^2} = \frac{1}{199} = \cdot0050251$$

$$''b = \frac{'b^2}{\left\{1 + \surd(1 - 'b^2)\right\}^2} = \cdot0000063313,$$

but $\frac{'b^2}{4} = \cdot0000063129$, which differs from $''b$ only in the ninth place of decimals; therefore $''b$ may be taken equal $\frac{'b^2}{4}$ without ſenſible error; and conſequently $^{(m)}b = {}'b$, or $m = 1$.

Now

$$b\left(\frac{b}{2} + \frac{1 + {}'b}{2}\right) = \frac{b^2}{4}(1 + {}'b) = \frac{b^2}{4} \times \frac{200}{199} = \cdot 005$$

$${}'P = 1 + {}'b = 1 + \frac{1}{199} = \frac{200}{199}$$

$${}'Q = \frac{b}{2} + \frac{b}{2} \cdot \frac{'b}{2} = \cdot 0707108$$

$$\text{hyp. log. } \frac{4}{'b} = \text{hyp. log. } 796 = 6{\cdot}679599$$

$$\frac{b}{2} \cdot {}'P \cdot {}'Q \cdot \text{hyp. log. } \frac{4}{'b} = \cdot 0334818.$$

Hence $1 - \cdot 005 + \cdot 0334818 = 1{\cdot}0284808$, and the whole perimeter $= 1{\cdot}0284808 \times 4 = 4{\cdot}1139232$, nearly.

X. QUESTION 100, *from the* CAMB. UN. CAL.

Shew that the rectification of the hyperbola may be deduced from that of the ellipſe?

FIRST SOLUTION.

Put 1 for the ſemi-tranſverſe axis, e for the excentricity, x the abſciſſa, reckoned from the centre on the tranſverſe axis, H the hyperbolic arc reckoned from the extremities of the ordinate and tranſverſe axis.

Then $\dot{H} = \dot{x}\sqrt{\left(\frac{e^2x^2 - 1}{x^2 - 1}\right)}$, and to aſſign the fluent of this expreſſion by means of elliptic arcs, put $x = \sqrt{\left(\frac{1 - \frac{z^2}{e^2}}{1 - z^2}\right)}$ and $\frac{1}{e} = m$; then

$$\sqrt{\left(\frac{e^2x^2 - 1}{x^2 - 1}\right)}\dot{x} = \frac{(1 - m^2)\dot{z}}{(1 - z^2)^{\frac{3}{2}}\sqrt{(1 - m^2z^2)}}.$$

Now, when $z = 0$, x is $= 1$, and when $z = 1$, x is $= \infty$; therefore the fluent of $\dot{x}\sqrt{\left(\frac{e^2x^2-1}{x^2-1}\right)}$ between the values of $x = 1$ and $x = \infty$ is equal to the fluent of $\frac{(1-m^2)\dot{z}}{(1-z^2)^{\frac{3}{2}}\sqrt{(1-m^2z^2)}}$ between the values of $z = 0$ and $z = 1$.

Put $z\sqrt{\left(\frac{1-m^2z^2}{1-z^2}\right)} = \text{F}$;

then taking the fluxions we have

$$\dot{\text{F}} = \dot{z}\sqrt{\left(\frac{1-m^2z^2}{1-z^2}\right)} - \frac{(1-m^2)\dot{z}}{\sqrt{\{(1-z^2)(1-m^2z^2)\}}} + \frac{(1-m^2)\dot{z}}{(1-z^2)^{\frac{3}{2}}(1-m^2z^2)^{\frac{1}{2}}}$$

therefore $\dot{\text{H}} = \dot{\text{F}} - \dot{z}\sqrt{\left(\frac{1-m^2z^2}{1-z^2}\right)} + \frac{(1-m^2)\dot{z}}{\sqrt{\{(1-z^2)(1-m^2z^2)}}$

But the fluent of $\dot{z}\sqrt{\left(\frac{1-m^2z^2}{1-z^2}\right)}$ is well known to be equal to the length of the elliptic arc whoſe ſemi-tranſverſe axis is 1, excentricity m and abſciſſa z.

Call this arc E. Then

$$\text{H} = \text{F} - \text{E} + \text{fluent of } \frac{(1-m^2)\dot{z}}{\sqrt{\{(1-z^2)(1-m^2z^2)\}}}.$$

To find the fluent of this laſt term put

$$m = \frac{1-\sqrt{(1-m'^2)}}{1+\sqrt{(1-m'^2)}} \text{ and } z = \frac{2}{1-m}.\dot{z}'\sqrt{\left(\frac{1-z'^2}{1-m'^2z'^2}\right)};$$

then the fluent of $\frac{(1-m^2)\dot{z}}{\sqrt{\{(1-z^2)(1-m^2z^2)\}}}$

$$= 2mz - \text{fluent of } 2(1+m)\dot{z}'\sqrt{\left(\frac{1-m'^2z'^2}{1-z'^2}\right)} \pm 2\text{E}.$$

But the fluent of $\dot{z}'\sqrt{\left(\frac{1-m'^2z'^2}{1-z'^2}\right)}$ is the elliptic arc E′ whoſe ſemi-tranſverſe axis is 1, excentricity m' and abſciſſa z'

$$\text{Hence H}\begin{cases} = \text{F} + 2mz - 2(1+m)\text{E}' + \text{E} \\ = z\sqrt{\left(\frac{1-m^2z^2}{1-z^2}\right)} + 2mz - 2(1+m)\text{E}' + \text{E}. \end{cases}$$

So that the length of an hyperbolic arc is aſſigned by means of two elliptic arcs and an algebraical quantity.

This diſcovery was firſt made by Mr. Landen, and publiſhed in the Philoſophical Tranſactions for 1775, and afterwards in his Memoirs*; but the ſubſtance of the above ſolution is taken from Mr. Woodhouse's Paper, referred to in the preceding ſolution.

SECOND SOLUTION, *by* M. C. BOSSUT.

Let 1 denote the ſemi-tranſverſe axis, and b the ſemi-conjugate axis of an ellipſe; and E an arch of this ellipſe intercepted between the conjugate axis and an ordinate parallel thereto at the diſtance x from the centre.

Then it is well known that $\dot{E} = \dot{x}\sqrt{\left\{\frac{1-(1-b^2)x^2}{1-x^2}\right\}}$.

Put $1-(1-b^2)x^2 = u^2$, then $x = \sqrt{\left(\frac{1-u^2}{1-b^2}\right)}$; and

$$E = \frac{u^2\dot{u}}{\sqrt{\{(1+b^2)u^2-u^4-b^2\}}}.\ \text{Let}\ \frac{\sqrt{\{(1+b^2)u^2-u^4-b^2\}}}{u}$$

$= y$, and the laſt equation will be transformed into

$$\dot{E} = -\frac{\dot{y}}{2} - \frac{(1+b^2-y^2)\dot{y}}{\sqrt{\{(1+b^2-y^2)^2-4b^2\}}},$$

which may be further abridged by putting $1+b^2 = c^2$, $1+b = \tau$, $1-c = \pi$; for then

$$\dot{E} = -\frac{\dot{y}}{2} - \frac{(c^2-y^2\dot{y}}{2\sqrt{(\tau^2-y^2)}\cdot\sqrt{(\pi^2-y^2)}}.$$

Let us now aſſume

$$\frac{c^2-y^2}{\sqrt{(\tau^2-y^2)}\cdot\sqrt{(\pi^2-y^2)}} = \frac{A\sqrt{(\tau^2-y^2)}}{\sqrt{(\pi^2-y^2)}} + \frac{B\sqrt{(\pi^2-y^2)}}{\sqrt{(\tau^2-y^2)}}.$$

By means of this equation $A+B = 1$, $A\tau^2 + B\pi^2 = c^2$, and conſequently $A = \frac{c^2-\pi^2}{\tau^2-\pi^2}$, $B = \frac{\tau^2-c^2}{\tau^2-\pi^2}$; ſo that

$$\dot{E} = -\frac{\dot{y}}{2} - \frac{A\dot{y}\sqrt{(\tau^2-y^2)}}{2\sqrt{(\pi^2-y^2)}} - \frac{B\dot{y}\sqrt{(\pi^2-y^2)}}{2\sqrt{(\tau^2-y^2)}}:\ \text{or}$$

$$E = -\frac{y}{2} - \frac{A}{2}\int\frac{\dot{y}\sqrt{(\tau^2-y^2)}}{\sqrt{(\pi^2-y^2)}} - \frac{B}{2}\int\frac{\dot{y}\sqrt{(\pi^2-y^2)}}{\tau^2-y^2)}.$$

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* Mr. Landen's Memoir is reprinted in the early numbers of the Old Series of the Repository.

In order to find the fluent of $\frac{\dot{y}\surd(\tau^2-y^2)}{\surd(\varpi^2-y^2)}$, put $\tau^2-y^2=z^2$;

then $\int\frac{\dot{y}\surd(\tau^2-y^2)}{\surd(\varpi^2-y^2)}=-\int\frac{z^2\dot{z}}{\surd(\tau^2-z^2)\,.\,\surd\{(z^2-(\tau^2-\varpi^2)\}}$,

which is known to represent the arch of an ellipſe intercepted between the conjugate axis and an ordinate parallel thereto, at the diſtance $\frac{\tau\surd(\tau^2-z^2)}{\surd(\tau^2-\varpi^2)}$ from the centre.

Call this arch E′.

And to find the fluent of $\frac{\dot{y}\surd(\varpi^2-y^2)}{\surd(\tau^2-y^2)}$, put $\varpi^2-y^2=t^2$;

then $\int\frac{\dot{y}\surd(\varpi^2-y^2)}{\surd(\tau^2-y^2)}=\int-\frac{t^2\dot{t}}{\surd(\varpi^2-t^2)\,.\,\surd(t^2+\tau^2-\varpi^2)}=$
$\frac{\surd(\varpi^2-t^2)\,.\,\surd(t^2+\tau^2-\varpi^2)}{t}-\int\frac{-\varpi^2(\tau^2-\varpi^2)\dot{t}}{t^2\surd(\varpi^2-t^2)\,.\,\surd(t^2+\tau^2-\varpi^2)}$.

But the fluent of $\frac{-\varpi^2(\tau^2-\varpi^2)\dot{t}}{t^2\surd(\varpi^2-t^2)\,.\,\surd(t^2+\tau^2-\varpi^2)}$ expreſſes the arch of an hyperbola of which the semi-axes are ϖ and $\surd(\tau^2-\varpi^2)$, and abſciſſa, reckoned from the centre, on the axis ϖ, $\frac{\varpi^2\surd(t^2+\tau^2-\varpi^2)}{\tau t}$.

Call this arch H.

Hence it appears, that

$$E=-\frac{y}{2}=\frac{A}{2}\times E'-\frac{B}{2}\times\frac{\surd(\varpi^2-t^2)\,.\,\surd(t^2+\tau^2-\varpi^2)}{t}+\frac{B}{2}\times H;$$

and, conſequently,

$$H=\frac{2E}{B}+\frac{y}{B}+\frac{A\times E'}{B}+\frac{\surd(\varpi^2-t^2)\,.\,\surd(t^2+\tau^2-\varpi^2)}{t}.$$

XI. QUESTION 101, *from the* CAMB. UN. CAL.

In the lemniſcata, whoſe equation is $(x^2+y^2)^2=x^2-y^2$, it is required to aſſign two arcs equal to one another.

First Solution, *by Mr.* Wallace, *R. M. College.*

1. Let CPA (fig. 171, pl. 9,) be the lemniſcata, C its centre, CA its ſemi-axis, and PB an ordinate at any point P in the curve; then CB $= x$, PB $= y$. Join CP and put r for the variable line CP, and a for the ſemi-axis CA, then, the given equation $(x^2 + y^2)^2 = x^2 - y^2$ may be expreſſed thus,

$$r^2 = a\sqrt{(x^2 - y^2)}.$$

Put φ for the variable angle PCA, then, becauſe $x = r\,\text{coſ.}\,\varphi$ and $y = r\,\text{ſin.}\,\varphi$, this laſt equation becomes $r^2 = ar\sqrt{(\text{coſ.}^2\,\varphi - \text{ſin.}^2\,\varphi)}$, which, by ſubſtituting coſ. 2φ for its equal $\text{coſ.}^2\,\varphi - \text{ſin.}^2\,\varphi$, and dividing by r becomes

$$r = a\sqrt{(\text{coſ.}\,2\varphi)} \ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

Draw Cp indefinitely near to CP, and pq perpendicular to CP, then, if z be put for AP, the arch of the curve, it is manifeſt that $\dot{z} = \sqrt{(\mathrm{P}q^2 + pq^2)}$.

But Pq is the fluxion of r, or of $a\sqrt{(\text{coſ.}\,2\varphi)}$, which is $\dfrac{-a\dot{\varphi}\,\text{ſin.}\,2\varphi}{\sqrt{(\text{coſ.}\,2\varphi)}}$, and pq is equal to $r\dot{\varphi}$, that is to $a\dot{\varphi}\sqrt{(\text{coſ.}\,2\varphi)}$, therefore ſubſtituting for Pq, and pq, theſe values and putting 1 for $\text{ſin.}^2\,2\varphi + \text{coſ.}^2\,2\varphi$, we have

$$\dot{z} = \frac{a\dot{\varphi}}{\sqrt{(\text{coſ.}\,2\varphi)}} \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

This fluxionary expreſſion, although very ſimple in its form, does not admit of an algebraic integral, nor even of one involving circular arcs or logarithms only. The integral may however be expreſſed in a finite form by means of two elliptic arcs; but the indefinite rectification of the curve is not the object of our preſent enquiry.

2. Leaving for the preſent the conſideration of the lemniſcata, let us ſuppoſe that φ and ψ are two angles ſo related to each other that the product of their co-ſines, or $\text{coſ.}\,\varphi\,\text{coſ.}\,\psi$ is a conſtant number, then, taking the fluxion of the product, we ſhall have

$$\dot{\varphi}\,\text{ſin.}\,\varphi\,\text{coſ.}\,\psi + \dot{\psi}\,\text{ſin.}\,\psi\,\text{coſ.}\,\varphi = 0.$$

Let us next ſuppoſe that the conſtant number is the coſine of 45°, or that

$$\text{coſ.}\,\varphi\,\text{coſ.}\,\psi = \sqrt{\tfrac{1}{2}}$$

then, by taking the ſquares of each ſide of this equation and putting $1 - \text{ſin.}^2\,\varphi$ for $\text{coſ.}^2\,\varphi$ we get

$$(1 - \text{ſin.}^2\,\varphi)\,\text{coſ.}\,\psi^2 = \tfrac{1}{2}$$

and $\frac{1}{2}(2\,\text{coſ.}^2\psi - 1) = \text{ſin.}^2\varphi\,\text{coſ.}^2\psi$:
but $2\,\text{coſ.}^2\psi - 1 = \text{coſ.}\,2\varphi$
therefore $\sqrt{(\frac{1}{2}\,\text{coſ.}\,2\varphi)} = \text{ſin.}\,\varphi\,\text{coſ.}\,\psi$,

and, ſince φ and ψ enter preciſely alike into the aſſumed equation $\text{coſ.}\,\varphi\,\text{coſ.}\,\psi = \sqrt{\frac{1}{2}}$ we have alſo

$$\sqrt{(\tfrac{1}{2}\,\text{coſ.}\,2\psi)} = \text{ſin.}\,\psi\,\text{coſ.}\,\varphi.$$

Let theſe values of $\text{ſin.}\,\varphi\,\text{coſ.}\,\psi$ and $\text{ſin.}\,\psi\,\text{coſ.}\,\varphi$ be ſubſtituted in the laſt fluxionary equation and it becomes

$$\dot\varphi\sqrt{(\tfrac{1}{2}\,\text{coſ.}\,2\psi)} + \dot\psi\sqrt{(\tfrac{1}{2}\,\text{coſ.}\,2\varphi)} = 0$$

and hence $\dfrac{\dot\varphi}{\sqrt{(\text{coſ.}\,2\varphi)}} + \dfrac{\dot\psi}{\sqrt{(\text{coſ.}\,2\psi)}} = 0 \;\ldots\ldots\ldots\ldots (3)$

Hence we learn that whatever values be given to the angles φ and ψ, provided the product of their coſines is $\sqrt{\frac{1}{2}}$, the ſum of the fluents of the two expreſſions $\dfrac{\dot\varphi}{\sqrt{(\text{coſ.}\,2\varphi)}}$ and $\dfrac{\dot\psi}{\sqrt{(\text{coſ.}\,2\psi)}}$, under whatever form they may be exhibited, or that ſum multiplied by any conſtant quantity, muſt neceſſarily be a conſtant quantity.

3. Let us now recur to the lemniſcata and ſuppoſe that CP, CQ, are two ſtraight lines drawn from the centre meeting the curve in P and Q, and that the angles PCA, QCA which we ſhall denote by φ and ψ reſpectively, are ſuch that the product of the coſines is equal to $\sqrt{\frac{1}{2}}$ or to the coſine of half a right angle, then as from equation (2) the fluxion of the arch AP will be $\dfrac{a\dot\varphi}{\sqrt{(\text{coſ.}\,2\varphi)}}$ and that of the arch AQ $\dfrac{a\dot\psi}{\sqrt{(\text{coſ.}\,2\psi)}}$ it is manifeſt that the ſum of the arches AP, AQ, will have the ſame property as the ſum of the fluents of the two quantities which conſtitute the fluxionary equation (3), that is, *the ſum of the arches AP, AQ, will be equal to a conſtant quantity.*

To determine this quantity we have only to give determinate values to the angles φ and ψ, and the conſtant quantity will evidently be the ſum of the arches of the lemniſcata correſponding to theſe values. Let us therefore aſſume $\psi = 0$, and conſequently $\text{coſ.}\,\psi = 1$, then from the equation $\text{coſ.}\,\varphi\,\text{coſ.}\,\psi = \text{coſ.}\,45^\circ$ we have $\varphi = 45^\circ$; now when $\psi = 0$, the arch of the curve correſponding to φ vaniſhes, and when $\varphi = 45^\circ$, it appears from the equation (1) which expreſſes the nature of the curve that the correſponding arch of the lemniſcata is its quadrant APC. Therefore the conſtant quantity to which the ſum of the arches AP, and AQ is always equal is the quadrant APQC. This property of the curve is very remarkable, and furniſhes directly a ſolution to the queſtion; for ſince the quadrant APQC is equal to the ſum of the arches AQ, AP, from each take away the common arch AP and the arch AQ reckoned from the one extremity of the ſemi-axis will be equal to the arch CP reckoned from the other extremity. This property of the curve which anſwers the queſtion may be ſtated in the form of a theorem: thus,

Let C be the centre and CA the ſemi-axis of the lemniſcata; from C let there be drawn two chords CP, CQ, meeting the curve in P and Q, and let the angles ACP, ACQ, be ſuch that the product of their coſines is equal to the coſine of half a right angle, then the arches of the curve, reckoned from the points P, Q, to the neareſt extremities of the axis, are equal to each other.

This theorem, which is analogous to thoſe relating to arches of conic ſections given in Art. 3, Part II. Vol. I. of the Repoſitory is capable of being rendered more general.

If we ſuppoſe the arches φ and ψ equal to each other, then the points P and Q will coincide at a point R; let RC be the poſition in which the lines CP, CQ, coincide and let θ denote the angle RCA, then as coſ. φ coſ. ψ becomes coſ.^{2} θ we have coſ.^{2} $\theta = \sqrt{\tfrac{1}{2}}$, therefore the angle θ is given, and conſequently the point R, that is the middle of the quadrantal arch CRQ, is given. Thus it appears that the quadrant of the lemniſcata like the quadrant of a circle admits of being biſected geometrically. It may alſo like the circle be divided into any number of equal parts by the reſolution of equations, but I ſhall not proſecute this ſubject at preſent any further.

SECOND SOLUTION, *by Mr.* IVORY, *R. M. College.*

The equation of the lemniſcata will be ſatisfied by putting

$$x^2 = \tfrac{1}{2}(z^2 + z^4)$$
$$y^2 = \tfrac{1}{2}(z^2 - z^4)$$

that is, by putting $x = \frac{z}{\sqrt{2}}\sqrt{(1 + z^2)}$

$$y = \frac{z}{\sqrt{2}}\sqrt{(1 - z^2)}$$

Let w be the arch of the curve correſponding to the co-ordinates x and y: then

$$\dot{w} = \sqrt{(\dot{x}^2 + \dot{y}^2)} = \frac{1}{\sqrt{2}} \times \dot{z}\sqrt{\left\{\frac{(1+2z^2)^2}{1+z^2} + \frac{(1-2z^2)^2}{1-z^2}\right\}} = \frac{\dot{z}}{\sqrt{(1-z^4)}}.$$

And if x' and y' be any other two co-ordinates, and w' the correſponding arch of the curve; and if

$$x' = \frac{z'}{\sqrt{2}} \quad (1 + z'^2)$$

$$y' = \frac{z'}{\sqrt{2}} \sqrt{(1 - z'^2)}$$

then in like manner $\dot{w}' = \dfrac{\dot{z}'}{\sqrt{(1 - z'^4)}}$.

Suppoſe now that z and z' are ſo related that

$z'^2 = \dfrac{1-z^2}{1+z^2}$ or $z' = \sqrt{\left(\dfrac{1-z^2}{1+z^2}\right)}$; then $\dot{z}' = \dfrac{-\dot{z}}{\sqrt{(1-z^4)}} \times \dfrac{2z}{1+z^2}$,

and $\sqrt{(1-z'^4)} = \dfrac{2z}{1+z^2}$: therefore $\dfrac{\dot{z}'}{\sqrt{(1-z'^4)}} = \dfrac{-\dot{z}}{\sqrt{(1-z^4)}}$;

that is, $\dot{w}' = -\dot{w}$, or $\dot{w}' + \dot{w} = 0$; and taking the fluents $w' + w = C$, where C is a conſtant quantity.

From the equation of the lemniſcata it is apparent, that while x^2 increaſes from 0 to 1, y^2 firſt increaſes to a maximum value, and then decreaſes to 0. Therefore taking all poſſible values of $+x$, between the limits 0 and $+1$, upon an axis, there will correſpond to each value of $+x$, a double ordinate $\pm y$, one on one ſide of the axis, and one on the other ſide of it: and thus an oval ſhaped figure will be completed on one ſide of the origin of x. And, in like manner, taking all poſſible values of $-x$, between the limits 0 and -1, upon the ſame axis, an equal and ſimilar oval ſhaped figure will be formed on the other ſide of the origin of x. Thus the lemniſcata conſiſts of two equal and ſimilar oval ſhaped curves, joined together in the origin of x, reſembling the numerical character 8, as the figure A*nm*B*μν*ACA (fig. 172, pl. 9.)

Let A$q = x$, $qn = y$; alſo A$p = x'$ and $pm = y'$: then $w =$ arch An, and $w' =$ arch Am: and the equation $w' + w = C$, ſhews that arch An + arch Am is a conſtant quantity. The relation between z and z' ſhews that $z' = 1$ when $z = 0$; and hence $x' = 1$, $y' = 0$, when $x = 0$ and $y = 0$: therefore the arch Am is equal to the ſemi-oval A*nm*B, when the arch An is equal to 0. Hence it is obvious that the ſemi-oval A*nm*B is equal to the conſtant quantity C. Therefore the two arcs An and Am are conſtantly equal to the ſemi-oval A*nm*B. Therefore the arch An is equal to the arch Bm: and thus an infinite number of equal arches may be aſſigned on the lemniſcata, reckoned from oppoſite ends of the ſemi-oval.

Since $z'^2 = \frac{1-z^2}{1+z^2}$, therefore $z^2 = \frac{1-z'^2}{1+z'^2}$; but $z^2 = x^2+y^2$, and $\frac{y'^2}{x'^2} = \frac{1-z'^2}{1+z'^2}$: therefore $x^2+y^2 = \frac{y'^2}{x'^2}$. Hence this geometrical conſtruction for finding equal arcs on the lemniſcata: draw BD perpendicular to AB and take BO = the chord An, then join AO cutting the curve in m; and the arch An is equal to the arch Bm.

THIRD SOLUTION, *by* P. Q. *of C. C. C.*

Put v for x^2+y^2, then by the queſtion v^2 will be equal to x^2-y^2, and conſequently

$$x = \frac{1}{\sqrt{2}}\sqrt{(v+v^2)}\,, \qquad \dot{x} = \frac{\dot{v}}{\sqrt{2}}\cdot\frac{\frac{1}{2}+v}{\sqrt{(v+v^2)}}$$

$$y = \frac{1}{\sqrt{2}}\sqrt{(v-v^2)}\,, \qquad \dot{y} = \frac{\dot{v}}{\sqrt{2}}\cdot\frac{\frac{1}{2}-v}{\sqrt{(v+v^2)}}$$

and $\sqrt{(\dot{x}^2+\dot{y}^2)} = \frac{\dot{v}}{2\sqrt{v}\sqrt{(1-v^2)}}$.

Now we are told by the very ingenious Mr. Landen, in the firſt Theorem of his 111 Table of Fluents given at the end of his Mathematical Memoirs, (ſee alſo vol. 1. p. 60, of the Mathematical Repoſitory, Old Series) that the fluent of $\frac{\dot{v}}{\sqrt{v}\sqrt{(1-v^2)}}$ generated whilſt v from 0 becomes equal to any quantity c is equal to the fluent of the ſame fluxion generated whilſt v from $\frac{1-c}{1+c}$ becomes equal to 1. From this property of the fluent we immediately derive a ſolution of the queſtion. For the figure of the lemniſcata being ſuppoſed known, let C (fig. 173, pl. 9.) be the point in which the curve interſects itſelf, and CA its ſemi-axis, which in the queſtion is ſuppoſed = 1, alſo CD and DP any abſciſſa and its correſponding ordinate, then the ſquare of the chord drawn from C to P will repreſent v: let us now ſuppoſe that another chord CP′ is drawn, and that the two chords CP, CP′, are ſo related to each other that CP being put equal to c, CP′ may be equal to $\sqrt{\left(\frac{1-c^2}{1+c^2}\right)}$, then it is obvious that the arches CP, P′A, may be conſidered as fluents generated according to the conditions of the theorem, they will therefore be equal to each other, and in this way may any number of equal arches be found agreeably to what is required in the queſtion.

It perhaps may not be amiſs to obſerve, that the firſt theorem of Mr. Landen's table (ſee as before his Memoirs, or the 1ſt. vol. of the Mathematical Repoſitory) exhibits the rectification of any arch of the lemniſcata by means of arches of two ellipſes, and the third and fourth theorems give the rectification of the whole curve; and it is particularly worthy of remark, that the whole curve may be rectified by means of the perimeters of the circle and of one ellipſe only, namely, that which has its excentricity equal to half its conjugate axis.

XII. QUESTION 162, *from the* CAMB. UN. CAL.

On a given ſtraight line, perpendicular to the horizon, deſcribe a circle, and draw a tangent at its higher extremity; if any point be aſſumed in this tangent, and a ſtraight line be drawn from it parallel to the diameter, cutting the circle in two points, and chords be drawn from thoſe points perpendicular to the diameter, the times of deſcribing the parts of this line, together with the times of deſcribing the correſponding chords with the velocities acquired at their extremities ſhall be equal?

FIRST SOLUTION, *by Mr.* BAZLEY.

Let AB (fig. 174. pl. 9.) be the vertical diameter of the circle AFBK; draw the tangent AG and GEF parallel to AB cutting the circle in E and F; draw the chords EI, FK, parallel to AG, cutting the diameter in C and D. Put $g = 16\frac{1}{12}$ feet; then by the laws of falling bodies, we have, the time of deſcribing GE or AC $= \sqrt{(AC \div g)}$, and the velocity acquired at C or E $= 2\sqrt{(g \times AC)}$. Now EI $= 2\sqrt{(AC \times CB)} = 2\sqrt{(AC \times AD)}$, and the time of deſcribing it, by uniform motion, is $2\sqrt{(AC \times AD)} \div 2\sqrt{(g \times AC)} = \sqrt{(AD \div g)}$, which is evidently the time of deſcribing AD; therefore the velocity at D or F $= 2\sqrt{(g \times AD)}$. Again, ſince FK $=$ EI $= 2\sqrt{(AC \times AD)}$, the time of deſcribing FK, by uniform motion, is $= 2\sqrt{(AC \times AD)} \div$

$2\sqrt{(g\times \mathrm{AD})} = \sqrt{(\mathrm{AC}\div g)}$, which is evidently the time of deſcribing AC. Hence the time of deſcribing AC or GE, together with the time of deſcribing EI is equal to the time of deſcribing AC, together with the time of deſcribing AD; that is, equal to the time of deſcribing AD or GF, together with the time of deſcribing FK.

SECOND SOLUTION, *by Mr.* W. MELLOR, *London.*

Put $\mathrm{EG} = x$, $\mathrm{FG} = y$, and $g = 16\frac{1}{12}$; then, by the property of the circle $\mathrm{EI} = \mathrm{FK} = 2\sqrt{(xy)}$. Now the time of deſcribing the diſtance x is $\sqrt{(x\div g)}$, and the velocity is then $= 2\sqrt{(gx)}$; wherefore the time of deſcribing EI with this velocity is $= \sqrt{(xy\div gx)} = \sqrt{(y\div g)}$; hence the whole time of deſcribing $\mathrm{EG} + \mathrm{EI}$ is $= \sqrt{(x\div g)} + \sqrt{(y\div g)}$. Again the time of deſcribing the diſtance y is $= \sqrt{(y\div g)}$, and the velocity is then $= 2\sqrt{(gy)}$. Wherefore, the time of deſcribing the chord FK with this velocity is $= \sqrt{(xy\div gy)} = \sqrt{(x\div g)}$. Or the whole time of deſcribing $\mathrm{GF} + \mathrm{FK}$ is $= \sqrt{(x\div g)} + \sqrt{(y\div g)}$; the very ſame as before.

XIII. QUESTION 103, *from the* CAMB. UN. CAL.

Let the compreſſive force be ſuppoſed, proportional to the ſquare of the denſity, and the force of gravity inverſely proportional to the ſquare of the diſtance from the earth's centre, it is required to find the law of denſity in the atmoſphere?

SOLUTION, *by Mr.* BAZLEY.

To render the ſolution as perſpicuous as may be, let C (fig. 175, pl. 9.) be the earth's centre, CA its radius, and conceive CAS to be the axis of a cylinder of the atmoſphere whoſe baſe is unity and its height indefinite. Put $\mathrm{CA} = r$, $\mathrm{CE} = x$, the denſity at $\mathrm{E} = y$, the denſity at $\mathrm{A} = d$, and let p denote the height of a column of air of the uniform denſity d, and weight equal to a column of mercury upon the ſame baſe $29\frac{1}{2}$ inches high.

Now let the curve BG be of ſuch a nature that the ordinate GE may repreſent the denſity at any point E; draw another ordinate IF indefinitely near to GE; and if two ſections of the cylinder perpendicular to CS be conceived to paſs through GEFI, it is plain that the quantity of air included between theſe ſections is $y\dot{x}$, that is, the magnitude into the denſity, and this drawn into the force of gravity at E gives $-\frac{r^2y\dot{x}}{x^2}$ for the fluxion of the weight or preſſure, which is negative, becauſe the weight or preſſure decreaſes as x increaſes. But by hypotheſis, the preſſure is as the ſquare of the denſity, and conſequently $-\frac{r^2y\dot{x}}{x^2} : 2y\dot{y}$ is a conſtant ratio, which, by the doctrine of fluxions, is the ſame as the ratio of their fluents; and, at the earth's ſurface A, this ratio is $pd : d^2$, that is as the weight to the ſquare of the denſity.

Hence, $pd : d^2 :: -\frac{r^2y\dot{x}}{x^2} : 2y\dot{y}$, and therefore $2p\dot{y} = -\frac{dr^2\dot{x}}{x^2}$; and the fluents are $2py = \frac{dr^2}{x}$: but when $y = d$, $x = r$, therefore, by correction, $2p\,(d - y) = dr - \frac{dr^2}{x}$; whence, $y = d - \frac{dr}{2p} + \frac{dr^2}{2px}$.

By following the ſame method, the law of denſity upon any other hypotheſis of gravity and compreſſion may be readily found:—Thus, if the denſity be as the compreſſion and the force of gravity uniform, we have, $pd : d :: -y\dot{x} : \dot{y}$, whence, $p\dot{y} = -y\dot{x}$, or $-\frac{p\dot{y}}{y} = \dot{x}$; and the fluents are $p \times$ hyp. log. $\frac{1}{y} = x$; and, by correction, $p \times$ hyp. log. $\frac{d}{y} = x - r$, which agrees with the rule uſually given by writers on pneumatics.

XIV. QUESTION 104, *from the* CAMB. UN. CAL.

A body revolving in a given circle, acted on by a force, which is inverſely proportional to the ſquare of the diſtance from the centre, in conſequence of an impulſe in the direction of its motion, begins to deſcribe an orbit of a given excentricity; find the velocity communicated to it by the impulſe?

FIRST SOLUTION, *by Mr.* BAZLEY.

Let S (fig. 176, pl. 9.) be the centre of force, A the point of impact. Since the direction of the body's motion at A is perpendicular to the axis AB, and its velocity (by the queſtion,) greater than that wherewith it deſcribed a circle at the ſame diſtance SA; it is plain, from the laws of central forces, that A is the lower apſe. Let F be the other focus, C the centre, V the velocity in the circle. Then, by Emerſon's centripetal forces, prop. 14, cor. 1, $\sqrt{(FA)} : \sqrt{(CA)}$ :: vel. in the ellipſe : vel. in the circle $= V$; hence, the velocity in the ellipſe at A is $= \sqrt{\{(FA \div CA) \times V\}}$; where FA and CA are given, becauſe SA and CS are given. Whence the velocity communicated is

$$\sqrt{\left(\frac{FA}{CA} \times V\right)} - V = \frac{\sqrt{(FA)} - \sqrt{(CA)}}{\sqrt{(CA)}} \times V.$$

SECOND SOLUTION, *by Mr.* R. SIMPSON.

Let S be the centre, and SA the radius of the given circle; then, ſince the body is acted on by a force which is inverſely proportional to the ſquare of the diſtance from the centre, the orbit deſcribed by the body muſt be a conic ſection. Let SF be taken on AS produced equal to twice the given excentricity, and make FB $=$ AS; then, becauſe the body, when at A, is impelled by a force acting perpendicularly to AS, it is obvious that AB will be the tranſverſe axis of the ſection, and A will be the place where the body is neareſt to the centre, and B the place where the body is at the greateſt diſtance from the centre. Now put v for the velocity in the circle at A, and V for that in the conic ſection, $r =$ the radius AS, and $p =$ the parameter, which is given, becauſe AS and the excentricity are given. Then, by prop. xvi. cor. 3, Principia,

$V : v :: \sqrt{(p)} : \sqrt{(2r)}$; or $V = v\sqrt{(p \div 2r)}$; and therefore $V - v = v\{\sqrt{(p \div 2r)} - 1\}$, the velocity required.

Or, if e be put for the excentricity, we have

$$V - v = v\left\{\sqrt{\left(\frac{2e - r}{e + r}\right)} - 1\right\}.$$

XV. QUESTION 105, *from the* CAMB. UN. CAL.

Two bodies whoſe weights are known, lying on a ſmooth horizontal plane, are connected by a flexible line, paſſing through a ſmall ring fixed at a given point between them; in this poſition a given velocity is communicated to one of them in a direction perpendicular to the line that joins their centres, and the other body is made to move directly towards the ring: inveſtigate the motion of the projected body, and find the angle deſcribed when the other body arrives at the ring.

FIRST SOLUTION, *by Mr.* CUNLIFFE.

Let A and B (fig. 177, pl. 9.) repreſent the two bodies at the commencement of motion, AB the line or ſtring connecting them paſſing through the ring at C. Let the body B, be that to which the given velocity of f feet per ſecond is communicated in a direction perpendicular to CB. Let the curve BB′ repreſent the track of the body B, the point b, another poſition of B, indefinitely near to B′: join CB′, Cb and with the centre C and radius CB′ deſcribe a ſmall arch B′d cutting Cb in d.

Put $CB = a$, $CB' = s$, then $db = \dot{s}$; alſo let z denote the arch of a circle (radius a) intercepted between CB, CB′. Let the velocity of B at B′ in the direction CB′ be denoted by v, and let the time of deſcribing BB′ be denoted by t; the weights of the bodies A and B being denoted by A and B reſpectively.

Then we ſhall obviouſly have $a : \dot{z} :: s : \frac{s\dot{z}}{a} = B'd$, and $db : B'd :: v : \frac{B'd \times v}{db} = \frac{vs\dot{z}}{a\dot{s}} =$ the velocity in the direction B′d; therefore $\frac{v^2 s\dot{z}^2}{a^2 \dot{s}^2}$ is the centrifugal force of the body B at B′.

Again, $db : B'b :: v : \frac{B'b \times v}{db} =$ the velocity of the body B in the direction of the curve B′b, and by the known property of tangents $B'b : B'd :: BC' : \frac{B'C \times B'd}{B'b} =$ the length of a perpendicular from C, upon a tangent to the curve at B′*a*. Now it is a well known law of a body revolving about a centre of force, that the velocities at different points in the orbit are inverſely as the lengths of perpendiculars from the centre upon tangents to thoſe points, hence we ſhall have $\frac{\text{B}'b \times v}{db} : f :: a : \frac{\text{BC}' \times \text{B}'d}{\text{B}'b}$; whence $\frac{\text{B}'\text{C} \times \text{B}'d \times v}{db} = fa$, that is $\frac{s^2\dot{z}v}{a\dot{s}} = fa$, by means of which the expreſſion for the centrifugal force before deduced, *viz.* $\frac{v^2s\dot{z}^2}{a^2\dot{s}^2}$ becomes $\frac{f^2a^2}{s^3}$.

Again, let F denote the force retarding the body B, in the direction B′C, ariſing from the inertia of the body A: then F × B will denote the motive force ariſing from the ſame cauſe in that direction, which being multiplied by the moment of time $\frac{\dot{s}}{v}$, gives $\text{F} \times \text{B} \times \frac{\dot{s}}{v}$, which must be equal to $\text{A}\dot{v}$ whence $\text{F} = \frac{\text{A}v\dot{v}}{\text{B}\dot{s}}$; therefore $\frac{f^2a^2}{s^3} - \text{F} = \frac{f^2a^2}{s^3} - \frac{\text{A}v\dot{v}}{\text{B}\dot{s}}$ will expreſs the accelerative force by which the body B is urged in the direction CB′. Then by the known laws of motion $\left(\frac{f^2a^2}{s^3} - \frac{\text{A}v\dot{v}}{\text{B}\dot{s}}\right) \times \dot{s} = v\dot{v}$ that is $\frac{f^2a^2\dot{s}}{s^3} - \frac{\text{A}v\dot{v}}{\text{B}} = v\dot{v}$, whence $\frac{f^2a^2\dot{s}}{s^3} = \left(\frac{\text{A}+\text{B}}{\text{B}}\right) \times v\dot{v}$; taking the fluents $-\frac{\frac{1}{2}f^2a^2}{s^2} = \left(\frac{\text{A}+\text{B}}{\text{B}}\right) \times \frac{v^2}{2}$ or $-\frac{f^2a^2}{s^2} = \left(\frac{\text{A}+\text{B}}{\text{B}}\right) \times v^2$, but when $s = a$, $v = 0$, therefore the correct fluent is $f^2 - \frac{f^2a^2}{s^2} = f^2 \times \left(\frac{s^2-a^2}{s^2}\right) = \left(\frac{\text{A}+\text{B}}{\text{B}}\right) \times v^2$, and by putting $\text{N} = \frac{\text{A}+\text{B}}{\text{B}}$ we ſhall have $v = \frac{f\sqrt{(s^2-a^2)}}{s\sqrt{(\text{N})}}$. Again, by a known theorem $t = \frac{\dot{s}}{v} = \frac{\text{N}^{\frac{1}{2}}s\dot{s}}{f\sqrt{(s^2-a^2)}}$, and taking the fluents $t = \frac{\text{N}^{\frac{1}{2}}\sqrt{(s^2-a^2)}}{f}$; alſo by what has been before deduced $\frac{s^2\dot{z}v}{a\dot{s}}$

$=fa$, whence $\dot{z}=\frac{\dot{f}a^2 s}{s^2 v}=\frac{N^{\frac{1}{2}}a^2\dot{s}}{s\sqrt{(s^2-a^2)}}$, and taking the fluents $z=N^{\frac{1}{2}}\times R$, where R denotes the length of a circular arch, radius a and ſecant s.

This latter expreſſion manifeſtly determines the angle deſcribed by the revolving body whilſt it acquires any aſſigned diſtance from the ring or centre C.

SECOND SOLUTION, *by Mr.* IVORY, *R. M. College.*

Let a denote the initial diſtance of the projected body from the ring, through which the connecting thread paſſes: alſo let ω be the weight (or maſs) of the projected body, and ω' that of the other body: and let b be the velocity of projection, or the ſpace deſcribed by the projected body in one ſecond of time, or in any other ſmall but finite portion of time. After any indefinite time τ, has elapſed, let ϱ denote the radius vector of the revolving body, and x, y, the rectangular, co-ordinates drawn from it on the ſtraight line from which the motion commences: then l denoting the whole length of the connecting ſtring, $l-\varrho$ will denote the diſtance of the body ω' from the ring; and $-\frac{\dot{\varrho}}{\dot{\tau}}$ will denote the velocity with which ω' approaches the ring; and $-\frac{\ddot{\varrho}}{\dot{\tau}^2}$ will be the accelerating force by which each particle of ω' is urged towards the ring; and $-\frac{\ddot{\varrho}}{\dot{\tau}^2}\times\omega'$ will denote the motive force by which the whole maſs of ω' is made to approach the ring. But the motive force of ω' is derived from the action of ω by means of the connecting ſtring; therefore, becauſe action and reaction are equal and contrary, the body ω is urged to the ring, or centre, by a motive force $=\frac{\ddot{\varrho}}{\dot{\tau}^2}\times\omega'$: therefore each particle of ω is urged by a centripetal force $=\frac{\ddot{\varrho}}{\dot{\tau}^2}\times\frac{\omega'}{\omega}$, for which the letter f may be written.

The centripetal force f acting in the direction of the radius vector may be reſolved into two other forces; *viz.* one $= \frac{x}{\rho} \times f$, acting parallel to x, and tending to diminiſh x; and another $= \frac{y}{\rho} \times f$, acting parallel to y, and tending to diminiſh y: but the forces tending to diminiſh x and y, are meaſured by $-\frac{\ddot{x}}{\dot{\tau}^2}$, and $-\frac{\ddot{y}}{\dot{\tau}^2}$; therefore, we have theſe two equations $-\frac{\ddot{x}}{\dot{\tau}^2} = \frac{x}{\rho} \times f$, and $-\frac{\ddot{y}}{\dot{\tau}^2} = \frac{y}{\rho} \times f$: or putting $\frac{\ddot{\rho}}{\dot{\tau}^2} \times \frac{\omega'}{\omega}$ for f,

$$\left.\begin{aligned} \frac{\ddot{x}}{\dot{\tau}^2} + \frac{\omega'}{\omega} \times \frac{x}{\rho} \times \frac{\ddot{\rho}}{\dot{\tau}^2} = 0 \\ \frac{\ddot{y}}{\dot{\tau}^2} + \frac{\omega'}{\omega} \times \frac{y}{\rho} \times \frac{\ddot{\rho}}{\dot{\tau}^2} = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let the firſt of theſe equations be multiplied by $2\dot{x}$, and the ſecond by $2\dot{y}$; then, adding them together, and remarking that $\rho^2 = x^2 + y^2$ and $2\rho\dot{\rho} = 2x\dot{x} + 2y\dot{y}$, we get

$$\frac{2\dot{x}\ddot{x} + 2\dot{y}\ddot{y}}{\dot{\tau}^2} + \frac{\omega'}{\omega} \times \frac{2\dot{\rho}\ddot{\rho}}{\dot{\tau}^2} = 0$$

and, taking the fluents,

$$\frac{\dot{x}^2 + \dot{y}^2}{\dot{\tau}^2} + \frac{\omega'}{\omega} \times \frac{\dot{\rho}^2}{\dot{\tau}^2} = c.$$

c being a conſtant quantity. To determine c, I remark that, at the commencement of the motion, or when $\tau = 0$, that $\frac{\dot{\rho}}{\dot{\tau}} = 0$; for the body is projected from reſt in a direction perpendicular to the radius vector, and, in the firſt inſtant of time, its velocity to or from the ring is $= 0$: therefore, when $\tau = 0$, $\frac{\dot{x}^2 + \dot{y}^2}{\dot{\tau}^2} = c$: but $\frac{\dot{x}^2 + \dot{y}^2}{\dot{\tau}^2}$ is manifeſtly the square of the velocity in the trajectory, and therefore $= b^2$, when $\tau = 0$: therefore $b^2 = c$: *The laſt equation*, then, is

$$\frac{\dot{x}^2 + \dot{y}^2}{\dot{\tau}^2} + \frac{\omega'}{\omega} \times \frac{\dot{\varrho}^2}{\dot{\tau}^2} = b^2 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Multiply the firſt of the equations (1) by x, and the ſecond by y; then having added them together, we get

$$\frac{x\ddot{x} + y\ddot{y}}{\dot{\tau}^2} + \frac{\omega'}{\omega} \times \frac{\varrho\ddot{\varrho}}{\dot{\tau}^2} = 0.$$

And this equation being added to the equation (2), there reſults

$$\frac{\dot{x}^2 + \dot{y}^2 + x\ddot{x} + y\ddot{y}}{\dot{\tau}} + \frac{\omega'}{\omega} \times \frac{\dot{\varrho}^2 + \varrho\ddot{\varrho}}{\dot{\tau}^2} = b.$$

But $\varrho\dot{\varrho} = x\dot{x} + y\dot{y}$ and $\dot{\varrho}^2 + \varrho\ddot{\varrho} = \dot{x}^2 + \dot{y}^2 + x\ddot{x} \pm \ddot{y}y$. therefore by ſubſtitution,

$$\frac{\dot{\varrho}^2 + \varrho\ddot{\varrho}}{\dot{\tau}} = \frac{\omega}{\omega + \omega'} \times b^2.$$

And, taking the fluents

$$\frac{\varrho\dot{\varrho}}{\dot{\tau}} = \frac{\omega}{\omega + \omega'} \times b^2\tau$$

no conſtant quantity being neceſſary becauſe $\frac{\dot{\varrho}}{\dot{\tau}} = 0$, when $\tau = 0$. And taking the fluents again

$$\varrho^2 = a^2 + \frac{\alpha b^2}{\omega + \omega'} \times \tau^2$$

$$\text{or } \varrho = a \times \sqrt{1 + \frac{\omega}{\omega + \omega'} \times \frac{b^2}{a^2}\tau^2}.$$

Again multiply the firſt of the equations (1) by y, and the ſecond by x, and ſubtract the one from the other, then

$$\frac{y\ddot{x} - x\ddot{y}}{\dot{\tau}^2} = 0.$$

And taking the fluents

$$\frac{y\dot{x} - x\dot{y}}{\dot{\tau}} = c$$

c being a conſtant quantity. To determine c, it is obvious that $y\dot{x} - x\dot{y}$ is the ſmall area deſcribed by the radius vector in the time $\dot{\tau}$, which area, when $\tau = 0$, is manifeſtly $= a \times b \times \dot{\tau}$.

Therefore $c = a \times b$: and

$$\frac{y\dot{x} - x\dot{y}}{\dot{\tau}} = ab,$$

Let φ = angle that ϱ makes with the line from which the motion commences (reckoned on the periphery of the circle whoſe radius $= 1$,) then $x = \varrho$ ſin. φ and $y = \varrho$ coſ. φ: And the laſt equation becomes, by ſubſtitution,

$$\varrho^2 \dot{\varphi} = ab\,\dot{\tau},$$

therefore $\dot{\varphi} = \frac{ab\dot{\tau}}{\varrho^2} = \frac{b}{a} \times \frac{\dot{\tau}}{1 + \frac{\omega}{\omega + \omega'} \times \frac{b^2}{a^2}\tau^2}$: hence

$$\varphi = \frac{\omega + \omega'}{\omega} \times \text{arc. tan. } \frac{b}{a}\tau\sqrt{\frac{\omega}{\omega + \omega'}}.$$

XVI. QUESTION 106, *from the* CAMB. UN. CAL.

If a body, ſuſpended by a ſtring, oſcillates through a quadrant (the extremity of the quadrant being the loweſt point,) the force ſtretching the ſtring at the loweſt point is three times that which is due to the weight of the body. Required proof?

FIRST SOLUTION, *by Mr.* BAZLEY.

Put r = radius of the quadrant, and $g = 16\frac{1}{12}$ feet; and if the force of gravity be meaſured by $2g$, we ſhall have the centrifugal force $= v^2 \div r$; but at the loweſt point of the quadrant the velocity v is $2\sqrt{(gr)}$, and the centrifugal force there $= v^2 \div r = 4g$. Hence $2g : 4g ::$ W, the weight of the body : 2W = the tenſion of the ſtring due to the centrifugal force; to which add the weight of the body itſelf, and we have the whole tenſion of the string $= 3$W.

SECOND SOLUTION, *by Mr.* JOHNSON, *Birmingham.*

Let C, (fig. 178, pl. 9.) be the point of ſuſpenſion, and ADB the quadrant deſcribed by the oſcillating body, AC being parallel, and BC perpendicular to the horizon. Then to find the force which ſtretches the ſtring at any point D, put r = AC, or the length of the ſtring; w = the weight of the body; $f = 32\frac{1}{6}$

feet, the force of gravity; $a =$ the angle ACD deſcribed by the body; $v =$ the velocity at D; and draw DE perpendicular to AC. Then, the force acting on the ſtring at D ariſes jointly from the centrifugal force in the direction DC, and the force of gravity in the ſame direction: but the centrifugal force at D is to the force of gravity as $v^2 \div r$ to 1, and the velocity in deſcending through the arc AD is equal to the velocity in deſcending through DE, or $v = \sqrt{(2g \cdot \mathrm{DE})} = \sqrt{(2gr\,\text{ſin.}\,a)}$; therefore, $v^2 \div r = 2g\,\text{ſin.}\,a$; or the centrifugal force is to the force of gravity as 2 ſin. a to 1; and, conſequently, the effect of the centrifugal force at D in the direction DC is equal to $w \times 2\,\text{ſin.}\,a$. Again, ſince the force of gravity acts in the direction ED, its effect in the direction DC will be equal to $w \times \text{ſin.}\,a$; and therefore the whole force ſtretching the ſtring will be equal to $w \times 3\,\text{ſin.}\,a$. But when the body arrives at B, ſin $a = 1$; and, conſequently, the whole force at the loweſt point is $= 3w$.

XVII. QUESTION 107, *from the* CAMB. UN. CAL.

Let two weights faſtened to two equal ſtrings, (l, l,) move in a vertical plane, the one oſcillating, the other revolving; and at the loweſt points let the velocities due to the heights h, h', be ſuch, that $\frac{h}{2l} = \frac{2l}{h'} = m^2$, then the time of the revolution of the one weight, is to the time of the oſcillation of the other, as m to 1. Required proof?

SOLUTION, *by Mr.* CUNLIFFE.

In the caſe of the revolving weight $\dfrac{l\dot{x}}{2g^{\frac{1}{2}}\sqrt{(h'-x)}\sqrt{(2lx-x^2)}}$ will expreſs the fluxion of the time; and in the caſe of the oſcillating weight $\dfrac{l\dot{v}}{2g^{\frac{1}{2}}\sqrt{(h-v)}\sqrt{(2lv-v^2)}}$ will expreſs the fluxion of the time of oſcillation. Now what is required is the ratio of the fluent of the former expreſſion, generated whilſt x from o becomes $2l$, to the fluent of the latter expreſſion, generated whilſt v from o, becomes h: or the ratio of the fluent $\frac{\dot{x}}{\sqrt{(h'-x)}\sqrt{(2lx-x^2)}}$ generated whilſt x from o, becomes $2l$, to the fluent of $\frac{\dot{v}}{\sqrt{(h-v)}\sqrt{(2lv-v^2)}}$ generated whilſt v from o becomes h.

From what is done at page 538, art. 460, vol. 2. of Simpſon's Fluxions, it appears that the fluent of $\frac{x}{\sqrt{(h'-x)}\sqrt{(2lx-x^2)}}$ when $x = 2l$, is

$$\frac{2p}{\sqrt{h'}} \times \left(1 + \frac{2l}{2.2.h'} + \frac{3.3.(2l)^2}{2.2.4.4.h'^2} + \frac{3.3.5.5.(2l)^3}{2.2.4.4.6.6.h'^3} + \&c.\right)$$

where p denotes the length of the quadrantal arc of a circle whoſe radius is 1.

Alſo the fluent of $\frac{\dot{v}}{\sqrt{(h-v)}\sqrt{(2lv-v^2)}}$ generated whilſt v from o becomes h is

$$\frac{2p}{\sqrt{(2l)}} \times \left(1 + \frac{h}{2.2.2l} + \frac{3.3.h^2}{2.2.4.4.(2l)^2} + \frac{3.3.5.5.h^3}{2.2.4.4.6.6.(2l)^3} + \&c.\right)$$

Now it is evident by mere inſpection, that if we put $\frac{h}{2l} = \frac{2l}{h'}$ the two ſeries will become identical, and therefore the ratio of the two fluents in the before mentioned circumſtances will be barely that of $\frac{2p}{\sqrt{h'}} : \frac{2p}{\sqrt{(2l)}} :: \sqrt{\frac{2l}{h'}} = m : 1$.

Q. E. D.

XVIII. QUESTION 108, *by Mr.* JAMES CUNLIFFE.

It is required to find two ſquare numbers, a^2, b^2, and a certain number n, which being either added to, or ſubtracted from either of the ſquares a^2, b^2, the ſums and differences thence ariſing ſhall all be rational ſquares.

SOLUTION.

SOLUTION, *by Mr.* CUNLIFFE, *the Propoſer.*

By the queſtion we ſhall have

$$a^2 + n = d^2$$
$$a^2 - n = e^2$$
$$b^2 + n = f^2$$
$$b^2 - n = g^2.$$

The ſum of the two firſt equations gives $2a^2 = d^2 + e^2$, and their difference $2n = d^2 - e^2$, or $n = \frac{d^2 - e^2}{2}$.

Whence it appears that e^2, a^2, and d^2, are three ſquares in arithmetical progreſſion, whoſe common difference is $n = \frac{d^2 - e^2}{2}$; wherefore we may put

$e = q^2 - p^2 + 2pq$, $a = p^2 + q^2$, and $d = p^2 - q^2 + 2pq$.
Whence $n = \frac{d^2 - e^2}{2} = 4pq\ (p^2 - q^2)$.

Again, by adding the third and fourth equations together we get $2b^2 = f^2 + g^2$; and taking their difference $2n = f^2 - g^2$, or $n = \frac{f^2 - g^2}{2}$.

Whence it appears that g^2, b^2, and f^2, are alſo three ſquares in arithmetical progreſſion whoſe common difference is $n = \frac{f^2 - g^2}{2}$; wherefore we may again put

$g = s^2 - r^2 + 2rs$, $b = r^2 + s^2$ and $f = r^2 - s^2 + 2rs$.
Whence $n = \frac{f^2 - g^2}{2} = 4rs\ (r^2 - s^2)$. Conſequently,

$4pq\ (p^2 - q^2) = 4rs\ (r^2 - s^2)$, or $pq\ (p^2 - q^2) = rs\ (r^2 - s^2)$.
Take $p = r$, and the preceding expreſſion will become

$rq(r^2 - q^2) = rs(r^2 - s^2)$; from whence $r^2 = \frac{q^3 - s^3}{q - s} = q^2 + qs + s^2$;

put $r = \frac{ts}{v} - q$, then $r^2 = \left(\frac{ts}{v} - q\right)^2 = \frac{t^2s^2}{v^2} - \frac{2tsq}{v} + q^2 = q^2 + qs + s^2$;

whence $s = \frac{q\ (v^2 + 2tv)}{t^2 - v^2}$: take $q = t^2 - v^2$, then $s = v^2 + 2tv$;

whence $p = r = \frac{ts}{v} - q = t^2 + tv + v^2$,

$$a = p^2 + q^2 = (t^2 + tv + v^2)^2 + (t^2 - v^2)^2,$$
$$b = r^2 + s^2 = (t^2 + tv + v^2)^2 + (v^2 + 2tv)^2,$$

and $n = 4pq\ (p^2 - q^2) = 4\ (t^2 - v^2)\ (t^2 + tv + v^2) \times \left\{ (t^2 + tv + v^2)^2 - (t^2 - v^2)^2 \right\}$.

where t and v may be expounded by any numbers at pleaſure.

Example. Take $t = 2$, and $v = 1$; then $a = 7^2 + 3^2 = 49 + 9 = 58$, or $a^2 = (58)^2 = 3364$; $b = 7^2 + 5^2 = 49 + 25 = 74$, or $b^2 = (74)^2 = 5476$; and $n = 12 \times 7 \times 40 = 3360$, and becauſe theſe values of a^2, b^2, and n, are all diviſible by the ſquare number 4, we may take $a^2 = 841$, $b^2 = 1369$, and $n = 840$.

Solutions were alſo received from Meſſrs. Bazley *and* A. M.

XIX. QUESTION 109, *by* MARLOVIENSIS.

In Counſellor Cooper's Letters on the Slave trade, we meet with the following queſtion, and an algebraic *formula* annexed, exhibiting a general ſolution thereto.

"It is required to ſhew the average annual import of negro ſlaves (and the whole number imported), having given the annual decrement, or proportion of deaths to the whole ſtock, and the number of exiſting ſtock at the end of a given term?

"General Solution.

$$\frac{ba^n}{(a - 1) \times (a^n - (a - 1)^n)} = \text{the average annual import.}$$

"Where n denotes the given interval in years.
b, the ſtock exiſting at the end of that term.
a, the reciprocal of the annual decrement.

"Example. The firſt export of negro ſlaves from Africa was in the year 1503, and in 1770, being an interval of 276 years, it was found, from a pretty exact eſtimate, that the number of negro ſlaves then exiſting in the European, &c. powers, was five millions: the proportion of deaths among them was alſo determined from a ſeries of obſervations to be about 1 in 20.

Theſe numbers ſubſtituted in the above formula, give
the average annual import 263,158
the total imported (in 276 year) .... 70,263,186 !"

And, we may add, if eſtimated to the preſent time, we behold near 80 millions of the human ſpecies, the innocent victims of avarice, cruelty, and wretchedneſs, by this diabolical traffic! an inveſtigation of the above *formula* is required?

FIRST SOLUTION, *by Mr.* LOWRY.

Put x for the number of ſlaves imported annually, $c = 1 - \frac{1}{a} = \frac{a-1}{a}$, (1 in a, or $\frac{1}{a}$, being the annual decrement);

Then

cx, will be the number remaining at the end of the firſt year;
$x + cx$, the augmented ſtock;
$cx + c^2 x$, the number remaining at the end of the ſecond year;
$x + cx + c^2 x$, the augmented ſtock;
$cx + c^2 x + c^3 x$, the number remaining at the end of the third year and ſo on. Conſequently,
$cx + c^2 x + c^3 x + \dots\dots\dots c^n x$, will be the number remaining at the end of n years.

Or, $cx\,(1 + c + c^2 + c^3 + \&c.\dots c^{n-1}) = b$:
but the ſum of the geometrical ſeries $1 + c + c^2 + c^3 + \dots c^{n-1}$ is $= \frac{1-(c)^{n-1}}{1-c}$.

Therefore $cx \times \left\{\frac{1-(c)^{n-1}}{1-c}\right\} = b$. Hence

$$x = \frac{b\,(1-c)}{c\left\{1-(c)^{n-1}\right\}} = \frac{ba^n}{(a-1)\times\left\{a^n-(a-1)^n\right\}},$$ the given formula.

SECOND SOLUTION, *by Mr.* BAZLEY.

This formula may be inveſtigated by induction, as follows. Put the average annual import $= x$; then by the nature of the queſtion, the ſtock exiſting at the end of the firſt year is

$b = x$

$b = x - \frac{x}{a}$, or $x = \frac{ba}{a-1} = \frac{ba}{(a-1)\times\left\{a-(a-1)\right\}}$.

Now for the ſecond year, the increaſed ſtock is

$x - \frac{x}{a} + x = \frac{2ax - x}{a}$,

and this decreaſed by the deaths is

$\frac{2ax - x}{a} - \frac{2ax - x}{a^2} = \frac{2a^2x - 3ax + x}{a^2} = b =$ ſtock

at the end of two years; hence we find

$$x = \frac{ba^2}{2a^2 - 3a + 1} = \frac{ba^2}{(a-1)\times\left\{a^2-(a-1)^2\right\}}.$$

Again, for the third year, the increaſed ſtock is

$\frac{2a^2x - 3ax + x}{a^2} + x = \frac{3a^2x - 3ax + x}{a^2}$,

and this decreaſed by the deaths as before, is

$\frac{3a^2x - 3ax + x}{a^2} - \frac{3a^2x - 3ax + x}{a^3} = \frac{3a^3x + 6a^2x + 4ax - x}{a^3}$

$= b =$ ſtock at the end of three years; and hence

$$x = \frac{ba^3}{3a^3 - 6a^2 + 4a - 1} = \frac{ba^3}{(a-1)\times\left\{a^3-(a-1)^3\right\}}.$$

And thus we may proceed for any number (n) of years and ſhall find,

$$x = \frac{ba^n}{(a-1)\times\left\{a^n-(a-1)^n\right\}}.$$

THIRD SOLUTION, *by* MARLOVIENSIS.

Let x denote the number of annual exports, $\frac{1}{a}$ $\left(\frac{1}{20}\right)$ the given decrement, n (276) the number of years in the given interval, and b (5000000) the exiſting ſtock at the end of the given term; then, by taking the decrement from the firſt annual export, and continually adding the ſame export, and deducting the decrement of each ſucceſſive year's ſtock as per queſt. we have (by reducing the ariſing terms,) the exiſting ſtock at the end of the 2st. 2d. 3d, 4th, &c. years reſpectively expreſſed by

$$\frac{(a-1)x}{a},\ (2a-1)\times\frac{(a-1)x}{a^2},\ (3a^2-3a+1)\times\frac{(a-1)x}{a^3},$$

$(4a^3-6a^2+4a-1)\times\frac{(a-1)x}{a^4}$, &c. and therefore generally for n years, by $na^{n-1}-\frac{n\,.\,n-1}{1\,.\,2}a^{n-2}+\frac{n\,.\,n-1\,.\,n-2}{1\,.\,2\,.\,3}\,.$ a^{n-3} (till it terminates) $\times$ into $\frac{(a-1)x}{a^n}$.

But this ſeries, continued to any aſſigned value of n, appears (by Clarke's Sum. of Series, p. 157,) to be equal to $a^n-(a-n)^n$; hence, we have

$$\left(a^n-(a-1)^n\right)\times\frac{(a-1)x}{a^n}, = b, \text{ from whence}$$

$$x=\frac{ba^n}{(a-1)\times\left(a^n-(a-1)^n\right)};$$ or, which is eaſier for computation, (n being here a large number,)

$$x=\frac{b}{a-1}\times\frac{1}{1-\left(\frac{a-1}{a}\right)^n}=$$

(by ſubſtituting the given numbers) 263158, the medium annual export; and therefore 70263186 the total number of Africans ſold for ſlaves to the year 1770.

XX. QUESTION 110, *by Mr.* Wm. Smith, *Liverpool.*

If a plane triangle be placed, ſo that two of its angular points may always be on two ſtraight lines given by poſition; it is required to determine the locus of the remaining angle?

First Solution, *by Mr.* Cunliffe.

Case 1. When the angular point is turned from the interſection of the two right lines given by poſition.

Let ACB (fig. 179. pl. 9.) be the given triangle, DA, DB the two right lines given by poſition; draw AE to meet BC in E, and make the angle AEB equal to the ſupplement of the angle ADB, and join DE.

Then becauſe the angles ADB, AEB are together equal to two right angles, a circle will paſs through the points A, E, B, D, and therefore the angle BDE is equal to the angle BAE, that is equal to a given angle; therefore the right line DE is given by poſition, and the point E will always be found in the right line DE.

Now BE is a right line of a given length intercepted between two right lines DB, DE, given by poſition, and C is another given point in the right line BE; therefore the locus of C is a given ellipſis, whoſe centre is D, as appears from Prop. 13, Book 2d. of De L'Hoſpital's Conic Sections.

Cor. 1. When the angle ACB is equal to the ſupplement of the angle ADB, the locus of C will be a right line.

Cor. 2. The locus of C will be a circle when BDE is a right angle, and C is the middle of BE; this is obvious from a known geneſis of the circle.

Case 2. When the angular point is turned towards the interſection of the two right lines given by poſition.

Produce BC, (fig. 180, pl. 9.) till the angle BEA be equal to the angle BDA, and draw the right line DE, then a circle will evidently paſs through the points A, E, D, B; and therefore the angle EDA is equal to the angle EBA, that is equal to a given angle; therefore DE is given by poſition, and BE is given in length. And for the very ſame reaſons as before, the locus of C will be a given ellipſis, having D for its centre.

Cor. 1. When the angle BCA is equal to the angle BDA, the locus of C will be a right line paſſing through D.

Cor. 2. When BDE is a right angle, and C biſects BE, the locus of C will be a circle whoſe centre is D, and diameter equal to BE, as appears from a known geneſis of the circle.

SECOND SOLUTION, *by Mr.* JOHNSON.

Let DX, DY, be the ſtraight lines given by poſition, and ABC the given triangle, having its angular points A, B, placed on the given ſtraight lines.

Draw DE to make the angle ADE equal to the given angle ABC, meeting BC produced if neceſſary in E, and join AE. Then becauſe the angle ADE is equal to the angle ABE, the points A, D, B, E, are in the circumference of a circle, and conſequently the angle AEB is equal to the angle ADB in fig. 180, or equal to its ſupplement in fig. 179. In either caſe AE is drawn to make a given angle with BC, and AC and the angle ACE are given in magnitude, therefore CE is alſo given in magnitude. It is therefore obvious that the ſtraight line DE will always paſs through the ſame given point E in the ſide BC, or in that ſide produced; or which is the ſame, the locus of the given point E is the ſtraight line DE. But DE is given in poſition becauſe it makes a given angle with DX, which is given in poſition; and DY is given in poſition, and BC, CE, are given in magnitude; therefore the locus of the point C is an ellipſe by the latter part of Mr. Ivory's Theorem (queſt. 88).

When the points C, E, coincide, it is evident that the locus of the point C is the ſtraight line DE; and when the angle BDE is a right angle, and BC = CE, the locus of C will be a circle; for in this caſe DC will be equal to BC = CE = to a given line.

XXI. QUESTION 111, *by Porus.*

Required to inſcribe, geometrically, the maximum ellipſis in a given trapezium?

SOLUTION, *by Mr. Lowry.*

Let ABCD (fig. 181, pl. 9.) be the given trapezium, O the centre, and POI a diameter of the required ellipſe, drawn from P, the point where the ellipſe touches the ſide AB. Produce the oppoſite ſides of the trapezium till they meet in G and H, and through I, the extremity of the diameter PI, draw the tangent FIL meeting BC in F, CD in E, and AH in L; then FL will be parallel to AB.

Let OK be the ſemi-conjugate to PI, and draw VON perpendicular to AB; then the rectangle $ON \times OK$ is equal to the rectangle of the ſemi-axes, (Prop. xxi. book iii. Robertſon's Conics,) and conſequently, when the ellipſe is a maximum, the rectangle $ON \times OK$ will be a maximum; or, ſince OI is equal to OP, ON will be equal to OV and VN to 2ON; and, becauſe the angle FBA is given, it is obvious that BF has to VN (Fn) or 2ON a given ratio; therefore when the ellipſe is a maximum the rectangle $BF \times OK$ will be a maximum.

Now the rectangles $BP \times FI$, $AP \times IL$, $PG \times IE$ are equal, each being equal to the ſquare of the ſemi-conjugate CK (Robertſon's Conics, p. ix. b. ii.); therefore

$PA : BP :: FI : IL$, and by compoſition
$AB : BP :: FL : IL$; but by ſimilar triangles
$BH : AB :: FH : FL$; therefore
$BH : BP :: FH : IL :: FH \times AP : IL \times AP$.

In a ſimilar way it is ſhewn that

$BP : BC :: IE \times PG : CF \times PG$:
but $IL \times AP = IE \times PG$; therefore
$BH : BC :: FH \times AP : CF \times PG$, or
$BH \times AP : BC \times AP :: FH \times AP : CF \times PG$:
but $CF \times PG = CF \times AP + CF \times AG$;
therefore, by permutation and converſion
$BH \times AP : BF \times AP :: BC \times AP : BF \times AP - CF \times AG$;
and again, by permutation and converſion
$BH : CH :: BF \times AP : CF \times AG$.

Find the point Q in AG produced ſuch, that $CH : BH :: AG : AQ$, then AQ will be given in magnitude, and the rectangle $BF \times AP$ will be equal to the rectangle $AQ \times CF$.

Let each of theſe rectangles be taken from the rectangle $BF \times AB$; then the rectangle $BF \times BP$ will be equal to the difference of the rectangles $BF \times AB$, $CF \times AQ$, that is equal to the difference of the rectangles $AB \times BC$, $BQ \times CF$, (becauſe $BC - BF$ is equal to CF).

On BC take RC ſuch, that the rectangle $RC \times BQ$ may be equal to the rectangle $AB \times BC$; then R will be a given point, and the difference of the rectangles $AB \times BC$, $BQ \times CF$, will be equal to the rectangle $RC \times RF$.

Therefore, the rectangle $BF \times BP$ is equal to the rectangle $RC \times RF$

or, $BP : RC :: RF : BF$;

but the rectangles $BP \times FI$, $AP \times IL$, being equal, FI will be to AP as IL to BP;

And by compoſition, and ſimilar triangles, as in the firſt part of the analyſis, we have

$FI : FH :: AP : BH$, or
$FI : FH :: AP \times BF = AQ \times CF : BH \times BF$;
but it is ſhewn above that
$BP : RC :: RF : BF$;
therefore, by compounding theſe two proportions, we have
$FI \times BP = (OK)^2 : FH \times RC :: AQ \times CF \times RF : BH \times (BF)^2$.

Now, ſince RC, AQ, and BH are given magnitudes, it is evident that the rectangle $OK \times BF$ will be a maximum when the ſolid $FH \times CF \times RF$ is a maximum; or if FS be drawn perpendicular to HF to meet a ſemicircle deſcribed on HR in S, then $FH \times RF$ will be equal to $(FS)^2$, and the ſolid $CF \times (FS)^2$ a maximum.

Let SX be a tangent to the circle at S, and draw ST to the centre T; then when $CF \times (FS)^2$ is a maximum, the ſub-tangent XT is equal to 2FC (ſee Simpſon on the Maxima et Minima):
But $2FC = 2FT - 2CT$; therefore $XT = 3FT - 2CT$; and $XT \times TF = (3FT - 2CT) \times FT = (ST)^2$, Euc. vi. 8 Cor.

Therefore TF may be found (Euc. vi. 29), and then the poſition of FL becomes known, and the ellipſe may be deſcribed by prop. xxvii. book iv. Robertſon's Conics.

Or, ſince BF and CF are known, and the rectangle $BF \times AP$ equal to the rectangle $AQ \times CF$; AP may be found and alſo BP, FI, and IL; then the ſemi-conjugate diameters OI, OK, will be given in magnitude and poſition to deſcribe the ellipſe, and the method of doing this is well known.

When the trapezium becomes a trapezoid the analyſis given above fails; and in this caſe if BFLA be the given trapezoid having the ſides FL, AB, parallel; then BF being given the ellipſe will be a maximum when the ſemi-diameter OK is a maximum, or when the rectangle $AP \times IL$ is a maximum, or (ſince BP has to IL a given ratio) when the rectangle $AP \times BP$ is a maximum, which is well known to be the caſe when AB is biſected in P, and then FL will be biſected in I, and the diameter PI will paſs through the point of interſection I.

When BFLA is a parallelogram, it is evident that the ſides will be biſected in the points of contact.

XXII. QUESTION 112, *by* PORUS.

If one end of a ſtring be fixed to the extremity of a body, placed on a perfectly poliſhed plane, and the other to a point in the plane, and any force be impreſſed upon the body, it is required to determine its motion?

SOLUTION,

SOLUTION, *by Mr.* LOWRY.

Fig. 182, pl. 9. Let q be the angle which the given plane makes with the horizontal plane, C the point to which the end of the ſtring is faſtened, and P the place of the body when it is at reſt, or is at the loweſt point of the plane to which the ſtring will ſuffer it to deſcend. Then, whatever force is impreſſed on the body, the curve deſcribed thereby muſt neceſſarily be a circle (or part of a circle) whoſe centre is C, and radius CP the length of the ſtring; and ſince the body is acted upon by a conſtant accelerative force, *viz.* that of gravity diminiſhed in the proportion of radius to the ſine of q, the motion of the body in the circle on the inclined plane will be analogous to that of a body moving in a circle placed on a vertical plane. This being premiſed, let the force impreſſed on the body at P be conſidered as acting perpendicularly to CP, ſince it may always be reduced to a force acting in that direction, and let V be the velocity communicated to the body at P by the impreſſed force. On the diameter Pp (produced if neceſſary) take DP equal to the diſtance which a body muſt deſcend on the plane to acquire the velocity V at the point P; then, if D be within the circle, let the chord FDG be drawn perpendicular to CP, and then F or G will be the point in the circle to which the body will aſcend before its velocity is entirely deſtroyed, and the body will continue to oſcillate in the arc FPG: but when D coincides with p, or the velocity acquired in deſcending through the diameter Dp is equal to V, the body will aſcend to p, and will continue to aſcend and deſcend in the ſemicircles PFp, PGp; and, laſtly, when D is in the diameter produced, the body will continue to revolve in the circle, and have the ſame velocity at p as would be acquired in deſcending through Dp.

To find the velocity (v) of the body at any point H in the circle, and the time (t) of deſcribing the arc FH, let HI be drawn perpendicular to CP, and put $g = 32\frac{1}{6}$ the force of gravity, $r =$ CP the length of the ſtring, $a =$ DP, $z =$ the arch FH and $x =$ PI; then, ſince the velocity acquired in deſcending through the arch FH is equal to that acquired in deſcending through DI, we have $v = \sqrt{(2g.\,\text{ſin}.\,q.\,\text{DI})} = \sqrt{\{2g.\,\text{ſin}.\,q.\,(a-x)\}}$, and therefore

$$\dot{t} = \frac{-\dot{z}}{v} = \frac{-\dot{z}}{\sqrt{\{2g.\text{ſin}.q.(a-x)\}}}\text{; or, ſince } \dot{z} = \frac{r\dot{x}}{\sqrt{(2rx-x^2)}},$$

$\dot{t}$ is =

$\dot{t}$ is $= \dfrac{-r\dot{x}}{\sqrt{\{2g.\ \text{ſin}.q.\ (a-x)\ (2rx-x^2)\}}}$; therefore,

$$t = \tfrac{1}{2}\sqrt{\frac{r}{g.\ \text{ſin}.q}} \times \int\left(\frac{-\dot{x}}{\sqrt{(ax-x^2)}} \times \frac{1}{\sqrt{(1-\frac{x}{2r}})}\right)$$

$$= \tfrac{1}{2}\sqrt{\frac{r}{g.\text{ſin}.q}} \times \int\left\{\frac{-\dot{x}}{\sqrt{(ax-x^2)}} \times (1+\tfrac{1}{2}.\frac{x}{2r}+\frac{1.3}{2.4}.\frac{x^2}{4r^2}+\&c.)\right\}$$

and the fluent from $x = 0$ to $x = a$ gives

$$t = \frac{p}{2}\sqrt{\frac{r}{g.\text{ſin}.q}} \times \left\{1+\frac{1^2}{2^2}.\frac{a}{2r}+\frac{1^2.3^2}{2^2.4^2}.\frac{a^2}{2^2r^2}+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}.\frac{a^3}{2^3.r^3}+\&c.\right\}$$

the time of a ſemi-oſcillation when D is within the circle, p being the length of a ſemi-circle whoſe radius is 1.

Alſo the fluent from $x = 0$ to $x = 2r$ gives

$$t = \frac{pr}{2}\sqrt{\frac{2}{a'g\ \text{ſin}.q}}\ .\ \left\{1+\frac{1^2}{2^2}.\frac{2r}{a'}+\frac{1^2.3^2}{2^2.4^2}.\frac{(2r)^2}{a'^2}+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}.\frac{(2r)^3}{a'^3}+\&c.\right\}$$

the time of a ſemi-revolution when D is without the circle, a' being equal to PD.

And when $a\ (a') = 2r$, we have

$$t = \frac{p}{2}\sqrt{\frac{r}{g.\ \text{ſin}.q}} \times \left\{1+(\tfrac{1}{2})^2+\left(\frac{1.3}{2.4}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2+\&c.\right\}$$

the time of aſcending through the ſemi-circle PFp.

If the force be impreſſed on the body at any given point H in the circle inſtead of at P, it is evident that a given velocity will be communicated to the body at H, and it will move either in the circle towards F, or in the direction Hh of the impreſſed force till it comes to the circle at h, when the velocity will be the ſame as at H. In either caſe the points F, f, to which the body will aſcend may be eaſily determined, and the circumſtances of the motion will then be preciſely the ſame as before.

XXIII. QUESTION 113, *by* SCOTICUS.

Let EF, a ſtraight line given in magnitude, be interpoſed between AC, BC, two ſtraight lines given in poſition ; let G be a given point in that line, then it is known that the locus of G is a given ellipſe. Draw ED, FD, perpendicular to the lines given by poſition meeting in D, and join DG, then DG ſhall be perpendicular to a tangent to the ellipſe at G. Required the demonſtration ?

FIRST

First Solution, *by Mr.* Lowry.

I ſhall firſt conſider the caſe in which the ſtraight lines AC, CB, are at right angles to each other, as this will aſſiſt us in demonſtrating the propoſition when the lines are placed in any other poſition.

Let the ellipſe which is the locus of the point G (fig. 183 and 184. pl. 10.) meet AC in A and CB in B; then AC will be equal to FG, and CB to EG, that is equal to the tranſverſe and conjugate axes. Let GD meet CA in H, and CB in I, and draw GP perpendicular to AC. Then by ſimilar triangles
AC (FG) : CB (EG) :: GD : GH, and
AC (FG) : CB (EG) :: GI : GD; therefore
$AC^2 : CB^2$ :: GI : GH :: CP : HP; and it follows from cor. 1. prop. xi. Emerſon's Conics, that HG is perpendicular to the tangent at G.

Now let the lines be placed in any other poſition (fig. 185, pl. 10.), and through the points F, C, E deſcribe a circle, and let the diameter OT be drawn through the point G; and draw CO, CT, to meet the ellipſe in I and K; then it is evident, from the latter part of my ſolution to the 88th queſtion, that CI (=TG) and CK (=CG) will be the ſemi-tranſverſe and ſemi-conjugate axes of the curve. Join CD, DO, DF; then becauſe the angles CFD, CED, are right angles, CD is a diameter of the circle, and conſequently DO is perpendicular to CD, and DT perpendicular to CT. Therefore between the ſtraight lines CO, CT, which are at right angles to each other, TO is interpoſed ſo that TG, GO, are reſpectively equal to the ſemi-axes CI, CK: and OD, TD, are drawn perpendicular to the lines CO, CT; conſequently DG is perpendicular to the tangent at G by the firſt part of the demonſtration above.

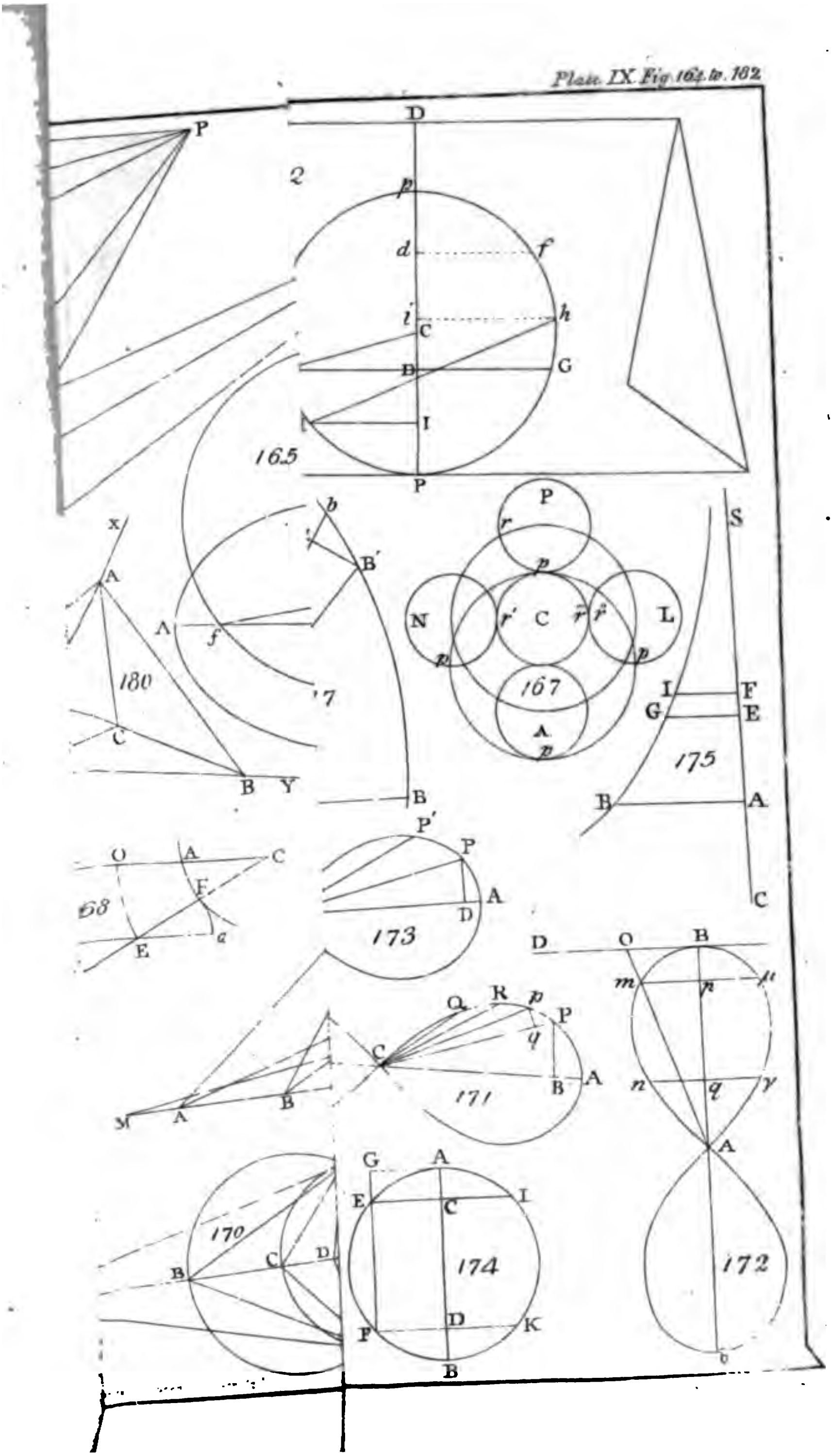
When the angle FCE is a right angle, and EF is biſected in G, the ellipſe becomes a circle whoſe radius is GC=FG=GE; and in this caſe the truth of the propoſition is evident from the nature of the circle.

The properties of the ellipſe which have been demonſtrated in this and the 88th queſtion may be elegantly applied to the ſolution of ſome uſeful ſtatical problems. We ſelect the following one as an example.

To place a given beam or rod loaded with a weight, ſo as to reſt in equilibrio when ſuſtained by two inclined planes given by poſition.

Let AC, CB, (fig. 186, pl. 10.) be the two inclined planes, making

Plate IX. Fig 164 to 182



making given angles with the horizontal plane XCY, EF the beam or rod reſting in equilibrio, and G the centre of gravity, or the place in which the weight is ſuppoſed to be concentrated. Then EG, FG, will be given in magnitude, and by the 88th queſtion the locus of the centre of gravity G, will be an ellipſe whoſe centre is C. Let this ellipſe be deſcribed; then (by a very obvious principle of ſtatics,) the beam or rod can only reſt in equilibrio when the centre of gravity is either at the higheſt or loweſt point to which it can poſſibly get; and it is evident in the preſent caſe that the centre of gravity muſt be at the higheſt point when an equilibrium takes place, for otherwiſe the beam or rod would not be at reſt, but its centre of gravity would continue to deſcend till it came to one of the inclined planes; and then the beam would lie wholly on that plane.

The point G, muſt therefore be that point in the ellipſe which is at the greateſt diſtance from the horizontal line XCY, or the point where a tangent parallel to that line touches the curve.

Draw ED, FD, perpendicular to the inclined planes AC, CB, and let DG be produced to meet XY in P. Then, by the propoſition which has been demonſtrated, DG will be perpendicular to the tangent at G, or DGP will be perpendicular to the horizontal line XCY to which the tangent is parallel; it is therefore manifeſt that if ED, FD, be drawn perpendicular to the inclined planes at the extremities of the beam or rod, and a line DP be drawn from the point of interſection, perpendicular to the horizontal plane, it will paſs through the centre of gravity of the beam or rod when it reſts in equilibrio.

This concluſion agrees with prop. 53, of Emerſon's Mechanics, and may ſerve as a confirmation of the truth of that propoſition deduced from the moſt unexceptionable principles.

Let KDL be drawn parallel to YX meeting the planes in K and L; then becauſe DG is perpendicular to KL and YX, the angle FDG is the complement of the angle FDK, or equal to the angle FKD, that is equal to the given angle FCY; and in the ſame manner it appears that the angle EDG is equal to the given angle ECX; therefore in the triangle FDE, the parts FG, EG, and the angles at the vertex made by DG are given. From whence the other parts of the triangle are readily found, and the poſition of the beam eaſily determined.

Mr. Samuel Clark has conſidered this problem at page 37 of his Introduction to the Theory and Practice of Mechanics, where he aſſerts that Mr. Emerſon's propoſition cannot poſſibly be true, unleſs in ſome very particular caſes there ſpecified.

Mr. Clark has, however, been too haſty in cenſuring Emerſon, and it may not be improper to ſhew, that the reſult in the two ſolutions is preciſely the ſame, when Clark's inveſtigation of the problem is completed.

Let $s =$ the ſine of the angle FCY,
$a =$ the ſine of the angle ECX,

$$\left.\begin{array}{l} m = \text{the ſine} \\ n = \text{the coſine} \end{array}\right\} \text{of the angle ECF,}$$

$p =$ FG, and $q =$ EG.

Then in the caſe of an equilibrium, Clark deduces this analogy.

FC : EC :: $qs + pna : pam^2 + nqs + pn^2a$; hence

$$\frac{\text{EC}}{\text{FC}} = \frac{\text{ſin. CFE}}{\text{ſin. CEF}} = \frac{pam^2 + nqs + pn^2a}{qs + pna} = \frac{nqs + pa}{qs + pna};$$

Therefore, by queſtion 74 of the Repoſitory, and reduction,

$$\frac{\text{coſ. CFE}}{\text{coſ. CEF}} = \frac{\text{ſin. DFE}}{\text{ſin. DEF}} = \frac{\text{DE}}{\text{DF}} = \frac{qs - qsn^2}{ap - pan^2} = \frac{qsm^2}{pam^2} = \frac{qs}{pa}:$$

$$\text{But } \frac{\text{DE}}{\text{DF}} = \frac{\text{ſin. FDG} \times q}{\text{ſin. EDG} \times p};$$

$$\text{therefore } \frac{qs}{pa} = \frac{\text{ſin. FDG} \times q}{\text{ſin. EDG} \times p};$$

$$\text{Or, } \frac{s}{a} = \frac{\text{ſin. FDG}}{\text{ſin. EDG}}.$$

Now the ſum of the angles FDG, EDG is equal to the ſum of the angles whoſe ſines are s and a, and, conſequently, the angle FDG is equal to the angle whoſe ſine is s, or equal to the angle FCY; and the angle EDG is equal to the angle whoſe ſine is a, or equal to the angle ECX; or if KDL be drawn parallel to YX, then the angle FDG is equal to the angle FKD, and the ſum of the angles FDK, FKD is a right angle; therefore the ſum of the angles FDK, FDG, or the angle CDK is a right angle, and conſequently DG is perpendicular to YCX.

The diameter G*g* being drawn, the point *g* will be the loweſt point in the ellipſe; and would be the place of the centre of gravity of the beam or rod when it was in equilibrio if its ends were conſtrained to move in the lines *cb*, *ck*; and the method of calculation in this caſe would be exactly the ſame as the preceding one.

SECOND SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

Fig. 187, pl. 10. Join CD, and draw CQ perpendicular to DG, and upon CQ take Cm = DG. The angles CQD, *CED, and* CFD being all right angles, a circle whoſe diameter *is CD will paſs through* the points C, Q, E, D, F: moreover the diameter CD is given, becauſe FE and the angle FCE are both given.

Biſect CD in O, and join OF, OG, and OE, then the iſoſceles triangle FOE is entirely given, as alſo the line OG.

By Prop 28, Book 2d, Emerſon's Geometry,
$(CG)^2 + (DG)^2 = (CG^2)^2 + (CM)^2 = 2(OC)^2 + 2(OG)^2$ = a given magnitude, becauſe CO, and OG are both given.

Again, ſince the points Q, E, D, F, C are in the circumference of the ſame circle.
$DG \times GQ = Cm \times GQ = FG \times GE$ = a given magnitude, becauſe FG and GE are both given.

Now we have found
$(CG)^2 + (Cm)^2$ = a given magnitude, and
$Cm \times GQ$ = a given magnitude,
which are known properties of an ellipſe, whoſe centre is C, and one pair of conjugate ſemi-diameters CG and C*m*; ſee Props. 34 and 37, Book 1ſt. of Emerſon's Conics.

Hence, C*m*, being the ſemi-diameter conjugate to CG, is parallel to a tangent at G, as is very well known; conſequently DG is perpendicular to a tangent to the ellipſe at G. *Q. E. D.*

The lengths of the principal ſemi-axes are $CO + OG$, and $CO - OG$; this is evident enough from what has been deduced, and the generation of the curve.

It is furthermore pretty obvious, that the ſum of the two principal axes of any ellipſe is leſs than the ſum of any other pair of conjugate diameters.

For CD is the ſum of the principal ſemi-axes which is leſs than $CG + DG$, the ſum of another pair of conjugate ſemi-diameters, becauſe any ſide of a plane triangle is leſs than the ſum of the other two.

The length and inclination of that pair of conjugate diameters whoſe ſum is a maximum, may be readily determined. In that caſe OG will be perpendicular to CD, and conſequently CG and DG equal to each other.

The angle of inclination of any two conjugate diameters, is the complement of QGC, which in the caſe laſt-mentioned becomes known.

XXIV. QUESTION 114, *by Mr.* IVORY, *R. M. College.*

Let AB, CD, be two ſtraight lines, cutting one another in G, and terminating in an ellipſe, of which F is one focus, and draw AF, FB, FC, FD; then will
$AF \times BG + BF \times AG : DG \times FC + FD \times GC :: AB : CD.$
Required the demonſtration?

SOLUTION.

SOLUTION, *by Mr.* LOWRY.

This property is general for all the conic ſections, and may be demonſtrated as follows.

Fig. 188. pl. 10. Let AI, BN, CM, DL and GK, be drawn perpendicular to the directrix of the ſection, and draw NG to meet IA produced if neceſſary in V. Then, by the general property of the ſections,

$AF : BF :: AI : BN$, and by ſimilar triangles
$GB : AG :: BN : AV$, (*); therefore
$AF \times GB : BF \times AG :: AI : AV$; and by compoſition
$AF.GB + BF.AG : BF.AG :: IV : AV :: IV.GB : AV.GB$.

But by ſimilar triangles,
$IV : GK :: VN : GN :: AB : GB$, or
$IV.GB = KG.AB$, and $AV.GB = BN.AG$ (*);
therefore, the proportion above becomes,
$AF.GB + BF.AG : KG.AB :: BF.AG : BN.AG :: BF : BN$.

And, in the ſame manner, it is ſhewn that,
$FC.DG + DF.CG : KG.CD :: AF : AI$, or $BF : BN$, (which is the ſame.) Therefore
$AF.GB + BF.AG : FC.DG + DF.CG :: KG.AB : KG.CD :: AB : CD$.

Mr. Bazley *alſo anſwered this Queſtion.*

XXV. QUESTION 115, *by Mr.* WALLACE, *R. M. College.*

Required the locus of the angles of a parallelogram, formed by drawing tangents at the vertices of any two conjugate diameters of a given ellipſe?

FIRST SOLUTION, *by* HYPATIA.

As an ellipſe may be conceived to be the orthographic projection of a circle, a parallelogram formed by tangents at the vertices of any two conjugate diameters of the ellipſe will be the orthographic projection of a ſquare deſcribed about the circle, and a ſtraight line drawn from the centre of the ellipſe to any angle of the parallelogram will be the orthographical projection of a ſtraight line drawn from the centre of the circle to the correſponding angle of the ſquare, and the former of theſe lines will have to the ſegment of it intercepted between the centre and the ellipſe the ſame ratio that the latter has to the radius of the circle, which ratio being given, namely that of $\sqrt{2}$ to 1, the locus of the angle of the parallelogram circumſcribing the ellipſe will evidently be a ſimilar concentric ellipſe, having its axes to thoſe of the other ellipſe in the ratio of $\sqrt{2}$ to 1.

SECOND SOLUTION, *by Mr.* LOWRY.

Let ABCD (fig. 189. pl. 10,) be a parallelogram formed by tangents, drawn at the extremities of any two conjugate diameters EG, FH, and draw the diagonals AC, DB, which biſect each other in O the centre of the ellipſe, as is too evident to need any demonſtration is this place. Let AC meet the ellipſe in *a* and *c*, and let BD meet the ellipſe in *b* and *d*, and draw FG meeting OB in V; then becauſe AF = FB and CG = BG, FG will be parallel to AC, and OV = VB = $\frac{1}{2}$OB; alſo FG is ordinately applied to the diameter O*b*. And, by Prop. vii. Book 2d, Robertſon's Conics

$\mathrm{OV}\ (\frac{1}{2}\mathrm{OB}) : \mathrm{O}b :: \mathrm{O}b : \mathrm{OB}$; therefore

$\mathrm{O}b^2 = \frac{1}{2}\mathrm{OB}^2$, or $2\mathrm{O}b^2 = \mathrm{OB}^2$; wherefore

$\mathrm{OB} : \mathrm{O}b\ (:: \mathrm{OD} : \mathrm{O}d) :: \sqrt{2} : 1.$

In the very ſame way it is ſhewn, that

$\mathrm{AO} : \mathrm{O}a\ (:: \mathrm{OC} : \mathrm{O}c) :: \sqrt{2} : 1.$

Therefore every ſemi-diameter in the required curve, has a conſtant ratio to the correſponding ſemi-diameter in the ellipſe; and conſequently the *locus* required is an ellipſe ſimilar and concentric to the given one.

Mr. Bazley *likewiſe anſwered this Queſtion.*

XXVI. QUESTION 116, *by* MARLOVIENSIS.

Suppoſe the river Thames, oppoſite *Canvey* Iſland, ſituated due eaſt and weſt, to be two miles broad at high-water, and one at low, and that the tide ebbs, at certain times, ſix hours, the breadth decreaſing from each ſide equal ſpaces in equal times. Suppoſe alſo the velocity of the water in the middle of the ſtream to increaſe as the time for the firſt three hours, when it then moves ſix knots *per* hour; and that the velocity in the middle is to that in any other part, as the ſemi-breadth of the water, to the neareſt diſtance of that part to the adjacent ſides. Theſe granted, what courſe, and how many knots *per* hour, muſt a boat ſail from the Kentiſh ſhore directly oppoſite *Bowley*, on the Eſſex ſide, at high water, to make *Scar Houſe*, on *Canvey* Iſland, after a three hours trip, which is exactly three miles to the eaſtward of the place ſailed from?

FIRST SOLUTION, *by* J. R. *of Mancheſter.*

By two approximating proceſſes, which immediately ariſe from the nature of the queſtion, it may be found, 1ſt, that when the veſſel arrives in the middle of the ſtream, which is in 1 hour 5-7ths, the tide will then have ſet her 1·88246 miles down the ſtream, which is due east: and, 2dly, from that time till ſhe comes to ground on the oppoſite ſhore, which is in 1 hour 2-7ths, it will have ſet her 2·89039 miles more to the eaſt. The ſum of theſe is 4·77285, the whole diſtance the tide has ſet her eaſt. If from this be taken 3 miles, the remainder is 1·77285, the weſting ſhe muſt make to gain the propoſed place. There are therefore given this, her weſting, and alſo her northing, 1·75 miles (the breadth from high water to half ebb) to find her courſe and diſtance. The firſt is north 45 deg. 22 min. weſterly, and the other 2·48998, which divided by 3, her time of ſailing, quotes ·82999 miles *per* hour, her rate of ſailing.

SECOND SOLUTION, *by the Propoſer* MARLOVIENSIS.

Let AB (fig. 190. pl. 10,) repreſent the breadth of the river at high water, B, Bowley, S, Scar Houſe, P, the middle of the ſtream, C the place of the veſſel at the end of any given time t from the commencement of her motion at A (which is at preſent ſuppoſed to be in the right line AB directly acroſs the river), and DA the decreaſe of PA at the ſame time.

From the conditions of the queſtion we have theſe analogies—

as 6 ho. : $\frac{1}{2}$ mile (decreaſe of PA) :: t ho. : $\frac{t}{12}$ = DA; and as 3 ho. : 6 knots, or miles :: t ho. : $2t$, the velocity of the water at P.

Put PC (the diſtance of the boat from the middle of the ſtream at the time t) $= x$, then will CD, its diſtance from the adjacent ſide, be expreſſed by $1 - \frac{t}{12} - x$, becauſe PD $= 1 - \frac{t}{12}$, and CD $=$ PD $-$ PC; and, *per* queſt. $1 - \frac{t}{12} : 1 - \frac{t}{12} - x :: 2t :$

$$\frac{\left(1 - \frac{t}{12} - x\right) 2t}{1 - \frac{t}{12}} = \frac{24t - 2t^2 - 24tx}{12 - t} = v,$$

a general expreſſion for the velocity of the current in miles *per* hour, at any given time t.

Again, ſince the veſſel is to ſail at an uniform rate, and to make ground on the oppoſite ſhore at the time of half ebb, we have

AO ($1\frac{3}{4}$ miles) : 3 ho. :: AP (1 m.) : $\frac{12}{7}$ ho. (which call a), the time ſhe comes to the middle of the ſtream, therefore, ſince a : AP (1) :: $t : \frac{t}{a} =$ AC, and PA $-$ AC $=$ PO, x will be truly expreſſed by $1 - \frac{t}{a} = \frac{a - t}{a}$, till ſhe arrives at P. And in like manner it appears that x will $= \frac{t - a}{a}$, while the veſſel ſails from P to the oppoſite ſhore at O. Hence, by ſubſtituting theſe values of x in the above general expreſſion for v, we have

$$\frac{24 - 2a}{a} \times \frac{t^2}{12 - t}, \text{ or, } \frac{12t^2}{12 - t} = v,$$

the velocity of the current till x from AP becomes 0; and

$$48 \cdot \frac{t}{12 - t} - 16 \cdot \frac{t^2}{12 - t} = v,$$

till x from 0 becomes PO, or ·75.

Let now y denote the drift, or diſtance the veſſel would be carried by the current from her ſuppoſed direct place at the end of the time t, and in the general fluxionary equation $\dot{y} = v\dot{t}$ (from principles well known) let the two latter values of v be written, and there will reſult from the former

$$\dot{y} = 12 \cdot \frac{t^2\dot{t}}{12 - t};$$ the fluent of which is,

while t from 0 becomes $\frac{12}{7}$,

$$12\,.(144 \text{ hyp. l. } \frac{12}{12-t} - \tfrac{1}{2}t^2 - 12t) =$$

in its ultimate value, 1·8826 $= y$ ($=$ PF), the diſtance the tide has ſet the veſſel down the ſtream, or ſo much due eaſt, when ſhe arrives at the middle of the water. And from the latter value of v, ariſes

$\dot{y} = 48\,.\,\frac{t\dot{t}}{12-t} - 16\,.\,\frac{t^2\dot{t}}{12-t}$, the fluent of which is (while t from $\frac{7}{12}$ (a) becomes 3) when corrected by conſidering when $y = 0$, t is $= a$,

$$48\,.\left(12 \text{ h. l. } \frac{12}{12-t} - t - (12.\text{ h.l. } \frac{12}{12-a} - a)\right)$$

$$-16\,.\left(144 \text{ h.l. } \frac{12}{12-t} - \tfrac{1}{2}t^2 - 12t - (144 \text{ h.l. } \frac{12}{12-a} - \tfrac{1}{2}a^2 - 12a)\right)$$

$=$ (in its final value) 2·889 $= y$ ($=$ NK), the diſtance the tide has ſet the veſſel farther eaſt when ſhe makes the oppoſite ſide of the river. The ſum of theſe is 4·7716 $=$ OK, the whole drift eaſterly. But, by the queſt. the veſſel is to arrive at I (S), which is only three miles to the eaſt of O (B); therefore, take OH $=$ (OK $-$ OI) IK $= 4·7716 - 3 = 1·7716$, and it is evidently the weſting ſhe muſt make to counteract the tide, and be three miles to the eaſt of the place ſhe ſailed from.

Hence, $\sqrt{(AO^2 + OH^2)} = AH = \sqrt{(1·75^2 + 1·7716^2)} =$ 2·4901 is the whole diſtance, which divided by 3, the time of ſailing, gives ·83 m. the uniform rate per hour; and her courſe ($\angle$BAH) north 45° 21′ weſterly. And had the boat's deſtination been the ſame diſtance to the *weſt* inſtead of the *eaſt* of the place ſailed from, her weſting would evidently have been 7·7716; and thence her diſtance 7·966, rate of ſailing 2·655 miles per hour, and courſe north 77° 19′ weſterly.

Rem. 1. This ſolution is manifeſtly founded on this eſtabliſhed principle—"a body acted upon by two forces at the ſame time in different directions, either equably or uniformly accelerated or retarded, will paſs through the ſame point at the end of any given time, as it would do if the two forces were each to act ſeparately and ſucceſſively for the ſaid time"—which affords a more direct proceſs in this caſe, than by inveſtigating the nature of the curve, or veſſel's real track ARI (ER being obviouſly $=$ PF) by means of an equation.

2. *The correction* of the fluent for finding the drift from the

middle of the river to the oppoſite ſide, would have been unneceſſary, had y and t been ſuppoſed to commence from o at F, and therefore $x = \frac{7t}{12}$, and the final value of t been taken $= \frac{9}{7} = 1$ ho. $\frac{2}{7}$, the time of ſailing from thence to K.

3. An approximate ſolution, independant of any fluxionary conſideration, may be obtained from the above firſt general expreſſion for v, by dividing the given interval (3 hours) into any number of parts, and finding the cotemporary values of t and x; the ſum of the reſulting values of v will exhibit the veſſel's drift from her *direct* place at the end of any aſſigned time t; in which, it is evident, the greater number of diviſions, or the ſmaller the intervals of time are, the nearer will the reſult approach to the truth.

XXVII. QUESTION 117, *by* HYPATIA.

Three circles being given by poſition, it is required to draw a tangent to each of them, ſo as to interſect in the ſame point, and have the ſegments between that point and the points of contact in a given ratio to each other?

SOLUTION, *by Mr.* LOWRY.

In order to ſimplify the ſolution of this problem as much as poſſible, we ſhall firſt find the locus of the point of interſection of tangents drawn to two circles given in magnitude and poſition, when theſe tangents have a given ratio.

Fig. 191, pl. 10. Let ID, HD, be two tangents interſecting in D; A and B the centres, and AI, BH, radii of the given circles drawn to the points of contact. Join A, D; B, D, and draw DV to make the angle VDI equal to the angle BDH; then the right-angled triangles IVD, HBD, will be ſimilar, and IV will be to BH, and VD to BD in the given ratio of ID to HD; therefore the ſquare of VD will have to the ſquare of BD a given ratio: let AC be taken to BC in this ratio, and join DC; through the point B let a circle be deſcribed to touch CD at D and meet

AD in E. and AB in F; join BE, and draw AL parallel to DC, meeting DB produced in L. Then becauſe CD is a tangent at D, the angle BFD is equal to the angle BDC (Euc. 32. iii.); that is, to the angle ALB (Euc. 29. i.); therefore the points E, B, L, A, are in the circumference of a circle: therefore

$ED . DA = BD . DL$: but by ſimilar triangles

$BC : BD :: AB : BL$, and by permutation and compoſition,

$BC : AC :: BD : DL :: (BD)^2 : DL \cdot BD$; and, by hypotheſis,

$BC : AC :: (BD)^2 : (VD)^2$; therefore

$DL \cdot DB \; (= ED \cdot DA) = (VD)^2$.

Now, (Euc. 47. 1.) $(VD)^2 - (IV)^2 = (ID)^2 = (AD)^2 - (IA)^2$; therefore

$(VD)^2 = (AD)^2 - \{(IA)^2 - (IV)^2\}$, and conſequently

$ED \cdot DA = (AD)^2 - \{(IA)^2 - (IV)^2\}$:

let each of theſe be taken from $(AD)^2$, and we have

$AE \cdot AD = (IA)^2 - (IV)^2$, or $AF \cdot AB = (IA)^2 - (IV)^2$:

but IA and IV are given in magnitude, therefore the rectangle $AF \cdot AB$ is given, and conſequently the point F is given. Now the points F, B, C, being given, the rectangle $BC \cdot CF$, or its equal the ſquare of DC (Euc. 36. iii.) will be given; therefore DC will be given, and the locus of the point D is a circle whoſe centre is C and radius CD.

The *conſtruction* as well as the *compoſition* is obvious from the analyſis given above; but it is neceſſary to remark that when ID is leſs than DH, the point C muſt be taken on AB produced beyond A; and when ID is greater than DH, the point C muſt be taken on AB produced beyond B: alſo when ID is equal to DH, the locus is a ſtraight line; for then DV is equal to DB, or $(DV)^2 = (DB)^2$, that is, $(AD)^2 - \{(IA)^2 - (IV)^2\} = (DB)^2$; therefore $(AD)^2 - (DB)^2 = (IA)^2 - (IV)^2$, a given magnitude, and the locus of D is a ſtraight line perpendicular to AB, as is well known.

If the circle which is the locus of the point D, meets AB in D, and two points P and R be taken in AC ſuch, that the rectangle $RC \cdot CP$ may be equal to the ſquare of the radius C*d*, and P*d* to R*d* in the given ratio of ID to HD; then if PD, RD, be drawn they will have the ſame ratio as the tangents ID, HD, (Playfair's Euc. prop. F. B. vi.) and P and R will be given points.

For by hypotheſis $CP : Cd :: Cd : RC$,
and by diviſion $Pd : Cd :: Rd : RC$,
and by permutation $Pd : Rd :: Cd : RC$:
but the ratio of Pd to Rd is given, and Cd is given, therefore RC is given, and conſequently CP is given; and therefore the points P and R are given. From hence we derive the following

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If two circles be given in magnitude and poſition, and tangents DI, DH, be drawn from any point to the circles ſo as to have a given ratio; two points P and R may be found ſuch, that the ſtraight lines PD, RD, being drawn from theſe points to the interſection of the tangents ſhall always have the ſame ratio as the tangents DI, DH.

A ſolution to the problem propoſed is immediately deducible from what has been done above. For let A, B, O, be the centres of the given circles, and ID, HD, KD, the tangents drawn as required; then, becauſe ID has to HD a given ratio, the locus of the point of interſection of theſe tangents will be a circle given in magnitude and poſition: alſo, becauſe DH has to DK a given ratio, the locus of the point of interſection of theſe tangents will likewiſe be a circle given in magnitude and poſition. Conſequently if theſe loci be deſcribed, as ſhewn in the analyſis, their interſection will give the point required.

XXVIII. QUESTION 118, *by* A. B.

Let C be a body of a given weight, (ſuppoſe 875 grains), placed on a ſmooth and poliſhed horizontal plane; let a thread or ſtring perfectly unelaſtic and inextenſible be attached to the body C, and after paſſing through two ſmall rings, A and B, (ſo as to act on the body in a direction parallel to the horizontal plane), fall down in a vertical direction: to the extremities of the thread, let two equal weights, P and Q (ſuppoſe each = 3500 grains), be faſtened; then, it is required to define the motion of the body C, and to aſcertain its place at the end of any given time, ſuppoſing it to begin to move from a given place, equally diſtant from the two rings, abſtracting from the weight of the ſtring, friction, and the reſiſtance of the air?

Solution, *by Mr.* Cunliffe, *R. M. College.*

Fig. 192, pl. 10. Join AB and draw CB perpendicular to it. Then the body C, being urged by two equal forces in the directions CB, CA, by the compoſition of forces, will move along the diagonal of the rhombus formed by the lines CB, CA; that is, the body C will move along the perpendicular CD, and the circumſtances of the motion will be preciſely the ſame as if only half the weight C was to be drawn along the ſmooth inflexible rod or line CD by the deſcent of the weight Q.

Put DB $= a$, BC $= x$, $g = 16\frac{1}{12}$ feet, the velocity with which Q deſcends $= v$, and the contemporaneous velocity of C $= u$. Then from what is deduced in the ſolution to the prize queſtion in the Gentleman's Diary for 1788, $\frac{\frac{1}{2}Cu\dot{u}}{v\dot{v}}$ will expreſs the inertia of $\frac{1}{2}$C in the direction CB or BQ, and therefore $Q + \frac{\frac{1}{2}Cu\dot{u}}{v\dot{v}}$ is the whole inertia in that direction.

Again $2gQ$ is the motive force by which the ſyſtem is urged in the direction CB, therefore $2gQ \div \left(Q + \frac{\frac{1}{2}Cu\dot{u}}{v\dot{v}}\right) = \frac{4gQv\dot{v}}{2Qv\dot{v} + Cu\dot{u}}$ expreſſes the accelerative force of the ſyſtem in the direction CB.

Then by a known theorem $\frac{-4gQv\dot{v}\dot{x}}{2Qv\dot{v} + Cu\dot{u}} = v\dot{v}$, whence $-4gQ\dot{x} = 2Qv\dot{v} + Cu\dot{u}$, and taking the fluents $8gQ\ (b - x) = 2Qv^2 + Cu^2$, where b denotes the value of BC at the commencement of the motion.

Again we ſhall obviously have $v^2 : u^2 :: \text{CD}^2 : \text{CB}^2$, whence $u^2 = v^2 \times \frac{\text{CB}^2}{\text{CD}^2} = \frac{v^2x^2}{x^2 - a^2}$, and therefore $8gQ \times (b - x) = 2Qv^2 + Cu^2 = 2Qv^2 + \frac{Cv^2x^2}{x^2 - a^2}$ which gives $v^2 =$

$$\frac{8gQ\,(b-x)\,(x^2-a^2)}{2Q\,(x^2-a^2)+Cx^2} = \frac{4gm\,(b-x)(x^2-a^2)}{x^2-ma^2},$$ by putting $m = \frac{2Q}{2Q+C}$, whence $v = \frac{2\sqrt{g}\sqrt{(b-x)}\,\sqrt{(x^2-a^2)}}{\sqrt{\left(\frac{x^2}{m}-a^2\right)}}$.

Moreover we ſhall have

$$\frac{-\dot{x}}{v} = -\dot{x}\sqrt{\frac{\frac{x^2}{m}-a^2}{x^2-a^2}} \div 2\sqrt{g}\,\sqrt{(b-x)}$$ for the fluxion of the time, the fluent of which is pretty obviously expreſſible by the time of a heavy body's deſcending freely, by the force of gravity, along the arch of a given hyperbola whoſe axis is perpendicular to the horizon.

But for the buſineſs of calculation it will be beſt to obtain an expreſſion for the time by means of an infinite ſeries.

For this purpoſe put $y = b - x$, or $x = b - y$, then

$$v = \frac{2\sqrt{(gm)}\sqrt{(b-x)}\sqrt{(x^2-a^2)}}{\sqrt{(x^2-ma^2)}} = 2\sqrt{(gmy)} \times \sqrt{\frac{(y-b)^2-a^2}{(y-b)^2-ma^2}}$$

whence $\dot{t} = \frac{-\dot{x}}{v} = \frac{\dot{y}}{v} = \dot{y}\,\frac{\sqrt{\frac{(y-b)^2-ma^2}{(y-b)^2-a^2}}}{2\sqrt{(gm)} \times y^{\frac{1}{2}}}$, the fluent of which may be readily found in an infinite ſeries, and by reverting this ſeries we may obtain the value of y in terms of t and given quantities.

XXIX. QUESTION 119, *by* PHILOMATHES OXONIENSIS.

It is required to find the fluent of $\dot{x}\sqrt{(1+\frac{1}{x^3})}$, generated, while x increaſes from $\frac{1}{10}$ to 10, and to give the work at large, bringing out the reſult true to four places of decimals, at leaſt?

SOLUTION.

SOLUTION, *by* PHILOMATHES OXONIENSIS, *the Propoſer.*

1. The given expreſſion is evidently $=\dot{x}\sqrt{\left(\frac{1+x^3}{x^3}\right)}=\dot{x}x^{-\frac{3}{2}}$ $\sqrt{(1+x^3)}$, the fluent of which may be aſſumed $=\sqrt{(1+x^3)}$ $\times(-2x^{-\frac{1}{2}}+\alpha x^{\frac{5}{2}}-\beta x^{\frac{11}{2}}+\gamma x^{\frac{17}{2}}-\delta x^{\frac{23}{2}}$, &c.); the fluxion of which expreſſion being taken, and put $=$ the given fluxion, we ſhall have

$$\dot{x}x^{-\frac{3}{2}}\sqrt{(1+x^3)}=\begin{cases}\dot{x}\sqrt{(1+x^3)}\times(x^{-\frac{3}{2}}+\frac{5}{2}\alpha x^{\frac{3}{2}}-\frac{11}{2}\beta x^{\frac{9}{2}}+\frac{17}{2}\gamma x^{\frac{15}{2}},\ \&c.)\\ +\frac{3\dot{x}x^2}{2\sqrt{(1+x^3)}}\times(-2x^{-\frac{1}{2}}+\alpha x^{\frac{5}{2}}-\beta x^{\frac{11}{2}}+\gamma x^{\frac{17}{2}},\&c.)\end{cases}$$

which equation being multiplied by $\frac{\sqrt{(1+x^3)}}{\dot{x}}$, and the like powers of x ranged one under the other, gives

$$\frac{1+x^3}{x^{\frac{3}{2}}}=\begin{cases}(1+x^3)\times(x^{-\frac{3}{2}}+\frac{5}{2}\alpha x^{\frac{3}{2}}-\frac{11}{2}\beta x^{\frac{9}{2}}+\frac{17}{2}\gamma x^{\frac{15}{2}}-\frac{23}{2}\delta x^{\frac{21}{2}},\ \&c.)\\ \qquad -3x^{\frac{3}{2}}+\frac{3}{2}\alpha x^{\frac{9}{2}}-\frac{3}{2}\beta x^{\frac{15}{2}}+\frac{3}{2}\gamma x^{\frac{21}{2}},\ \&c.\end{cases}$$

And this equation, when multiplication has been actually performed in the firſt line of it, will become

$$x^{-\frac{3}{2}}+x^{\frac{3}{2}}=\begin{cases}x^{-\frac{3}{2}}+\frac{5}{2}\alpha x^{\frac{3}{2}}-\frac{11}{2}\beta x^{\frac{9}{2}}+\frac{17}{2}\gamma x^{\frac{15}{2}}-\frac{23}{2}\delta x^{\frac{21}{2}},\&c.\\ +\quad x^{\frac{3}{2}}+\frac{5}{2}\alpha x^{\frac{9}{2}}-\frac{11}{2}\beta x^{\frac{15}{2}}+\frac{17}{2}\gamma x^{\frac{21}{2}},\ \&c.\\ -\quad 3x^{\frac{3}{2}}+\frac{3}{2}\alpha x^{\frac{9}{2}}-\frac{3}{2}\beta x^{\frac{15}{2}}+\frac{3}{2}\gamma x^{\frac{21}{2}},\ \&c.\end{cases}$$

from which we have

$$5\alpha=6,\quad 11\beta=8\alpha,\quad 17\gamma=14\beta,\quad 23\delta=20\gamma,\quad \&c.$$

and $\alpha=\frac{6}{5}$, $\quad\beta=\frac{8\alpha}{11}$, $\quad\gamma=\frac{14\beta}{17}$, $\quad\delta=\frac{20\gamma}{23}$, &c.

ſo that the variable part of the fluent ſought is

$$\sqrt{(1+x^3)}\times(-2x^{-\frac{1}{2}}+\frac{6}{5}x^{\frac{5}{2}}-\frac{6.8}{5.11}x^{\frac{11}{2}}+\frac{6.8.14}{5.11.17}x^{\frac{17}{2}}-\frac{6.8.14.20}{5.11.17.23}x^{\frac{23}{2}}\&c)$$

Now, although this ſeries wants a correction, and will not converge when x becomes greater than 1, ſtill it is very uſeful. For, when $x = \frac{1}{10}$, this expreſſion becomes

$$\sqrt{(1{\cdot}001)} \times \left(-\frac{2\sqrt{10}}{1} + \frac{6}{5.100\sqrt{10}} - \frac{6{\cdot}8}{5.11.100000\sqrt{10}}, \text{ \&c.} \right);$$

when $x = 1$, this expreſſion becomes

$$\sqrt{2} \times \left(-2 + \frac{6}{5} - \frac{6\,.\,8}{5\,.\,11} + \frac{6\,.\,8\,.\,14}{5\,.\,11\,.\,17}, \text{ \&c.} \right);$$

and the difference of theſe two quantities, viz.

$$\sqrt{(1{\cdot}001)} \times \left(\frac{2\sqrt{10}}{1} - \frac{6}{5.100\sqrt{10}} + \frac{6.8}{5.11.100000\sqrt{10}}, \text{ \&c.} \right);$$

$$-\sqrt{2} \times \left(2 - \frac{6}{5} + \frac{6\,.\,8}{5\,.\,11} - \frac{6\,.\,8\,.\,14}{5\,.\,11\,.\,17}, \text{ \&c.} \right),$$

is the quantity generated while x increaſes from $\frac{1}{10}$ to 1. The former of theſe ſeries converges very ſwiftly, and the latter may eaſily be computed.

2. We now want a convenient expreſſion of the part of the fluent generated while x increaſes from 1 to 10; which is eaſily attainable as follows:

Aſſume the fluent of $\dot{x}\sqrt{(1+x^{-3})} = \sqrt{(1+x^{-3})} \times (x - ax^{-2} + bx^{-5} - cx^{-8} + dx^{-11}$, &c.), and put the fluxion of this expreſſion = the given fluxion, and there will be

$$\dot{x}\sqrt{(1+x^{-3})} = \begin{cases} \dot{x}\sqrt{(1+x^{-3})} \times (1 + 2ax^{-3} - 5bx^{-6} + 8cx^{-9} - 11dx^{-12}, \text{\&c}) \\ \dfrac{-3\dot{x}}{2x^4\sqrt{(1+x^{-3})}} \times (x - ax^{-2} + bx^{-5} - cx^{-8} + dx^{-11}, \text{\&c}) \end{cases}$$

and this equation multiplied by $\frac{\sqrt{(1+x^{-3})}}{\dot{x}}$, and properly arranged, becomes

$$1+x^{-3} = \begin{cases} (1+x^{-3}) \times (1 + 2ax^{-3} - 5bx^{-6} + 8cx^{-9} - 11dx^{-12}, \text{ \&c.} \\ \qquad -\frac{3}{2}x^{-3} + \frac{3}{2}ax^{-6} - \frac{3}{2}bx^{-9} + \frac{3}{2}cx^{-12}, \text{ \&c.}; \end{cases}$$

that is, by actual multiplication,

$$1+x^{-3}=\left\{\begin{array}{l} 1+2ax^{-3}-5bx^{-6}+8cx^{-9}-11dx^{-12}, \text{ \&c.} \\ + \quad {}^{-3} \quad +2ax^{-6}-5bx^{-9}+8cx^{-12}, \text{ \&c.} \\ -\frac{3}{2}x^{-3}+\frac{3}{2}ax^{-6}-\frac{3}{2}bx^{-9}+\frac{3}{2}cx^{-12}, \text{ \&c.} \end{array}\right.$$

and hence we have

$$2a=\frac{3}{2},\ 5b=\frac{7}{2}a, 8c=\frac{13}{2}b, 11d=\frac{19}{2}c, \text{ \&c.}$$

$$\text{and } a=\frac{3}{4},\ b=\frac{7}{10}a, c=\frac{13}{6}b,\quad d=\frac{19}{22}c, \text{ \&c.}$$

We therefore now have another expreſſion of the variable part of the fluent ſought, viz:

$$\sqrt{(1+x^{-3})}.(x\ \frac{3}{4}x^{-2}+\frac{3\cdot7}{4\cdot10}x^{-5}-\frac{3\cdot7\cdot13}{4.10.16}x^{-8}+\frac{3\cdot7\cdot13\cdot19}{4.10.16.22}x^{-11}, \text{ \&c.}$$

which ſeries will converge while x increaſes from 1 to 10, and will ſerve our purpoſe. For,

when $x=1$, this expreſſion becomes

$$\sqrt{2}\times(1-\frac{3}{4}+\frac{3\cdot7}{4\cdot10}-\frac{3\cdot7\cdot13}{4\cdot10\cdot16}+\frac{3\cdot7\cdot13\cdot19}{4.10.16.22}, \text{ \&c.});$$

when $x=10$, this expreſſion becomes

$$\sqrt{(1\cdot001)}\times(10-\frac{3}{4\cdot100}+\frac{3\cdot7}{4\cdot10.100000}, \text{ \&c.});$$

and the difference of theſe two quantities, viz.

$$\sqrt{(1\cdot001)}\times(10-\frac{3}{4\cdot100}+\frac{3\cdot7}{4\cdot10.100000}, \text{ \&c.}$$

$$-\sqrt{2}\times(1-\frac{3}{4}+\frac{3\cdot7}{4\cdot10.}-\frac{3\cdot7\cdot13}{4.10.16}+\frac{3\cdot7\cdot13\cdot19}{4.10.16.22}, \text{ \&c.})$$

is the quantity generated while x increaſes from 1 to 10.

3. All that is now wanting to a full anſwer to this queſtion is the decimal value of the numerical ſeries obtained in the two preceding articles; and theſe may be computed as follows:

Computation of the values of the two ſwiftly converging ſeries.

$$\begin{array}{rcr} 2\sqrt{10} &=& 6{\cdot}324555, \\ +\dfrac{6{\cdot}8}{5{\cdot}11{\cdot}100000\sqrt{10}} &=& \underline{0{\cdot}000003,} \\ && +\,6{\cdot}324558, \\ -\dfrac{6}{5{\cdot}100\sqrt{10}} &=& \underline{-\,0{\cdot}370095,} \\ && 6{\cdot}320763, \\ \text{Value of three terms} && 9{\cdot}992505. \end{array} \qquad \begin{array}{rcr} 10 &=& 10{\cdot}000000, \\ +\dfrac{3{\cdot}7}{4{\cdot}10{\cdot}100000} &=& \underline{0{\cdot}000005,} \\ && +\,10{\cdot}000005, \\ -\dfrac{3}{4{\cdot}100} &=& \underline{-\,0{\cdot}007500} \\ \text{Value of three terms} && 9{\cdot}992505, \end{array}$$

Sum $16{\cdot}313268 \times \sqrt{(1{\cdot}001)} = 16{\cdot}321425$, the value of the two ſwiftly converging ſeries when multiplied by their common factor.

Computation of the values of the two ſlowly converging ſeries.

The firſt thirteen terms of the ſeries $2 - \frac{6}{5} + \frac{6\,.\,8}{5\,.\,11}$, &c. expreſſed in decimals continued to ſix places of figures, will be $2{\cdot}000000 - 1{\cdot}200000 + 0{\cdot}872727 - 0{\cdot}718717 + 0{\cdot}624971 - 0{\cdot}560319 + 0{\cdot}512292 - 0{\cdot}474807 + 0{\cdot}444500 - 0{\cdot}419340 + 0{\cdot}398017 - 0{\cdot}379647 + 0{\cdot}363606$. The ſum of the firſt ſix terms is 1·018662, and if the remaining terms be transformed into a more ſwiftly converging ſeries, by the method deſcribed in *Simpſon*'s *Diſſertations*, p. 62, we ſhall have $D^{I} = 0{\cdot}037485$, $D^{II} = 0{\cdot}007178$, $D^{III} = 0{\cdot}002031$, $D^{IV} = 0{\cdot}000721$, $D^{V} = 0{\cdot}000295$, and $D_{VI} = 0{\cdot}000129$; and the terms of the more ſwiftly converging ſeries will be

$$\begin{array}{rl} \frac{1}{2} \times 0{\cdot}512293 &= 0{\cdot}256146, \\ \frac{1}{4}D^{I} &= 0{\cdot}009371, \\ \frac{1}{8}D^{II} &= 0{\cdot}000897, \\ \frac{1}{16}D^{III} &= 0{\cdot}000127, \\ \frac{1}{32}D^{IV} &= 0{\cdot}000023, \\ \frac{1}{64}D^{V} &= 0{\cdot}000005, \\ \frac{1}{128}D^{VI} &= \underline{0{\cdot}000001,} \end{array}$$

$$\begin{array}{ll} \text{Value of this ſeries} \;\ldots\ldots & 0{\cdot}266570, \\ \text{Six terms above computed} & \underline{1{\cdot}018662,} \\ & 1{\cdot}285232, \quad \text{value of the original ſeries.} \end{array}$$

The firſt thirteen terms of the ſeries $1 - \frac{3}{4} + \frac{3 \cdot 7}{4 \cdot 10}$, &c. expreſſed in decimals as above, will be 1·000000 — 0·750000 + 0·525000 — 0·426563 + 0·368395 — 0·328924 + 0·299901 — 0·277409 + 0·259317 — 0·244356 + 0·231717 — 0·220855 + 0·211390. The ſum of the firſt ſix terms is 0·387908; and if the remaining terms be transformed into a more ſwiftly converging ſeries, by the method above mentioned, we ſhall have $D^{I} = 0·022492$, $D^{II} = 0·004400$, $D^{III} = 0·001269$, $D^{IV} = 0.000460$, $D^{V} = 0·000196$, $D^{VI} = 0·000097$; and the terms of the equivalent ſeries will be

$$\begin{aligned}
\tfrac{1}{2} \times 0·299901 &= 0·149951, \\
\tfrac{1}{4}D^{I} &= 0·005623, \\
\tfrac{1}{8}D^{II} &= 0·000550, \\
\tfrac{1}{16}D^{III} &= 0·000079, \\
\tfrac{1}{32}D^{IV} &= 0·000014, \\
\tfrac{1}{64}D^{V} &= 0·000003, \\
\tfrac{1}{128}D^{VI} &= 0·000001;
\end{aligned}$$

Value of this ſeries .. . 0·156221
Six terms above computed 0·387908

0·544129, the value of the original ſeries.
1·285232, the value of the other ſlowly [converging ſeries.

Sum $1·829361 \times \sqrt{2} = 2·587107$, the value of the two ſlowly converging ſeries when multiplied by their common factor.

We have now only to take 2·587107 the value of the two negative ſeries, from 16·321425, the value of the two affirmative ones, and there will remain 13.734318, the value of the fluent required.

XXX. PRIZE QUESTION 120, *by* EDINBURGENSIS.

A polygon being given in magnitude and poſition, it is required to deſcribe a polygon of an equal number of ſides, having its ſeveral angles on the ſeveral ſides of the given polygon, and ſuch that the perimeter ſhall be the leaſt poſſible?

SOLUTION, *by Mr.* LOWRY, *R. M. College.*

PROPOSITION. A LEMMA. (Fig. 193. pl. 10.)

Let ABCD . . . . .E be a given polygon, and PQRS . . . a polygon of an equal number of ſides inſcribed in, or havir ſeveral angles placed on the ſeveral ſides of, the given poly; then the perimeter of the inſcribed polygon will be the leaſt ; ble, when each two of its adjoining ſides make equal angles the corresponding ſide of the circumſcribing polygon; th when the angle TPA is equal to the angle QPB, the angle I to the angle RQC, the angle QRC to the angle SRD, th gle RSD to the angle TSE, the angle STE to the angle P and ſo on.

For let pQRST . . . . . .p be any other polygon inſcribe the given one, and let two of its ſides, as Tp, Qp make une angles with the ſide AB at p. Draw TIK perpendicula AB, meeting QP produced in K and join KP. Then bec the angle TPI is equal to the angle QPB, that is, to the a IPK, the right angled triangles TPI, KPI will be equal in e reſpect, therefore PT is equal to PK and IT to IK, and cc quently in the right angled triangles ITp, IKp, the ſides Tp will be equal; therefore KQ is equal to the ſum of the TP, QP, and $Kp + pQ$ equal to the ſum of the ſides Tp, but KQ is leſs than $Kp + pQ$, and therefore the ſum of ſides TP, QP is leſs than the ſum of the ſides Tp, pQ, and ſequently the perimeter of the polygon PQRST . . . . . .P is than the perimeter of the polygon pQRST . . . . . .p. the ſame may be proved, if any other two adjoining make unequal angles with the correſponding ſide of the circ ſcribing polygon; and therefore the perimeter of the poly PQRST . . . . . .P muſt be leſs than the perimeter of any c polygon that can be inſcribed in the given one.

This being demonſtrated, it remains only to determine the ſition of the inſcribed polygon, ſo that each two of its adjoi ſides may make equal angles with the correſponding ſid the given polygon; and to effect this in as perſpicuo manner as poſſible, I ſhall conſider the problem in diſtinct c according as the polygon conſiſts of an even or of an odd n ber of ſides.

CASE I.

When the nnmber of the ſides of the polygon is odd. Fig. 194, pl. 10.

Let ABCDE, &c. be the given polygon, and $a\,b\,c\,d\,e$, &c. the inſcribed polygon whoſe ſides make equal angles with the correſponding ſides of the given one, and let the angles of the polygon ABCDE, &c. be denoted by the letters placed at the angular points, *viz.* the angle EAB by A, the angle ABC by B, and ſo on. Alſo let the angle Aae, or Bab be denoted by a, the angle Bba, or Cbc by b, and ſo on; and let n be put for two right angles. Then, becauſe the ſum of the three angles in each of the triangles Bab, Cbc, &c. is equal to two right angles, we have

$$\mathrm{B}+a+b=\mathrm{C}+b+c=\mathrm{D}+c+d=\mathrm{E}+d+e=\mathrm{A}+e+a,\&\text{c.}=n,$$

and taking the ſum of the alternate terms

$$\mathrm{B}+a+b+\mathrm{D}+c+d+\mathrm{A}+e+a=\mathrm{C}+b+c+\mathrm{E}+d+e+n,$$

$$\text{or } 2a=n-\mathrm{B}+\mathrm{C}=\mathrm{D}+\mathrm{E}-\mathrm{A},\ \&\text{c.};$$

and in a ſimilar way we find

$$2b=n-\mathrm{C}+\mathrm{D}-\mathrm{E}+\mathrm{A}-\mathrm{B},\ \&\text{c.}$$

$$2c=n-\mathrm{D}+\mathrm{E}-\mathrm{A}+\mathrm{B}-\mathrm{C},\ \&\text{c.}$$

$$2d=n-\mathrm{E}+\mathrm{A}-\mathrm{B}+\mathrm{C}-\mathrm{D},\ \&\text{c.}$$

$$2e=n-\mathrm{E}-\mathrm{A}+\mathrm{B}-\mathrm{C}+\mathrm{D},\ \&\text{c.}$$

The angles being found, we have only to determine the poſition of one of the angular points a of the inſcribed polygon, and to do this, let the ſides ab, bc, cd, &c. be produced till the diſtances bi, cm, dp, es, af, &c. are equal to each other, and of any aſſumed magnitude. And let the perpendiculars ikl, mno, pqr, $ſtu$, fgh, &c. be drawn to the ſides of the given polygon, meeting the ſides of the inſcribed one in l, o, r, u, h, &c. Then, becauſe of the equal angles at a, b, c, d, e, &c. it is evident that ik will be equal to kl, mn to no, pq to qr, $ſt$ to tu, fg to gh; alſo, bi will be equal to bl, cm to co, dp to dr, es to eu, and af to ah, &c.

Now let P be any point in AB, and let PQ, QR, RS, ST, TU, &c. be drawn parallel to the ſides ab, bc, cd, de, ea, &c. reſpectively, and let aY be perpendicular to BC, and bZ perpendicular to AB produced. Then, becauſe $a\mathrm{B}\,.\,b\mathrm{Z}=\mathrm{B}b\,.\,a\mathrm{Y}$, we have $a\mathrm{B} : \mathrm{B}b :: a\mathrm{Y} : b\mathrm{Z}$: but by ſimilar triangles $ab : a\mathrm{Y} :: bi : ik$, and $ab : bz :: af\ (ah) : gh$; therefore $a\mathrm{Y} : b\mathrm{Z} :: ik : gh$, or $a\mathrm{B} : \mathrm{B}b :: ik : gh = fg$, or, becauſe PQ, ab are parallel,

aP

$$aP : bQ :: ik : gh = fg\,;$$

and in the ſame way it is ſhewn that

$$\begin{aligned} bQ &: cR :: nm : kl = ik \\ cR &: dS :: pq : on = nm \\ dS &: eT :: st : nq = pq, \text{ \&c to} \\ eT &: aU :: fg : ut = st\,; \end{aligned}$$

And by compounding theſe proportions, it is obvious that we ſhall finally obtain

$$aP : aU :: fg : fg :$$

Therefore aP is equal to aU. Whence it follows that whereverthe point P is taken in AB, the diſtance UP, will always be biſected in a; and from hence we derive the following ſimple and general conſtruction.

Aſſume any point P in one of the ſides, as AB, and draw PQ, QR, RS, ST, TU, &c. to make the given angles with the ſides AB, BC, &c. as determined in the analyſis, biſect U P in a, and a will be the poſition of one of the angular points of the required polygon.

If $2n + 1$ be the number of the ſides of the polygon, it is obvious from comparing the values of the angles as determined above, that the polygon can only be inſcribed in the manner ſpecified in the propoſition, when the ſum of $n + 1$ of the angles is leſs than the ſum of the n remaining angles, together with two right angles.

From the point U, let Ua' be drawn parallel to ab, $a'b'$ to bc, $b'c'$ to cd, $c'd'$ to de, $d'e'$ to ea, &c. then it is evident from above that ae' will be equal to a U, that is, equal to a P; therefore the points P and e' coincide. From whence we may draw the following curious and remarkable inference, *viz.* that beginning at any point P in one of the ſides AB, and drawing the lines PQ, QR, RS, &c. parallel to the ſides of the inſcribed polygon and continuing them twice round the figure, the laſt ſide de' will terminate in the point P, where we originally began.

The perimeter of the figure made by going twice round in the manner juſt deſcribed, will be equal to twice the perimeter of the inſcribed polygon $a\,b\,c\,d\,e$, &c. For ſince a P is equal to a U, Qb will be equal to ba', and conſequently $PQ + Ua'$ will be equal to twice ba; and in the ſame way it appears that $QR + a'b'$ is equal to twice bc, $RS + b'c'$ to twice cd, and ſo on; therefore the perimeter of the figure PQRSTU $a'b'c'd'$ P is equal to twice the perimeter of the polygon $a\,b\,c\,d\,e$, &c

In the triangle ABC (fig. 195, pl. 10.) we have $2a = n - B + C - A = 2C$, or $a = C$, and in the ſame way we obtain $b = A$, and $c = B$; hence the angles which the ſides of the inſcribed triangle make with the ſides of the given triangle are equal to the angles oppoſite thoſe ſides.

It is alſo obvious that each of the angles of the given triangle muſt be leſs than a right angle.

If the lines aC, bA, cB be drawn, they will be perpendicular to the ſides AB, BC, AC, reſpectively, as is too evident to need demonſtrating here.

CASE II.

When the given polygon conſiſts of an even number of ſides. (Fig. 196, pl. 10).

Denoting, as before, the angles of the given polygon by A, B, C, &c. and the angles made by the ſides of the inſcribed polygon and the ſides AB, BC, CD, &c. of the given one by a, b, c, d, &c. we have.

$B + a + b = C + b + c = D + c + d$, &c. .... $=$ to $f + a + A$, therefore

$B + a + b + D + c + d + F + e + f$ &c. $= C + b + c + E + d + e + f + a + A$, &c. or $B + D + F +$ &c. $= C + E + A +$ &c :

which ſhews that the ſides of the polygon $abcdef$, &c. can only make equal angles with the correſponding ſides of the given one, when the ſum of the angles B, D, F, &c. is equal to the ſum of the angles C, E, A, &c. which will always be the caſe when the given polygon can either be inſcribed in a circle, or is equiangular to one that can be ſo inſcribed.

It ſhews alſo, that the angles a, b, c, &c. are not determinable from the angles of the given polygon as in the caſe of the odd ſided figures, but that their values depend on the relations of the ſides of the polygon.

To determine the angles, let perpendiculars BQ, BI be drawn from any of the angular points as B, to the ſides CD and AF, and continued till LQ be equal to BL and IP to IB: alſo from the points Q and P, let perpendiculars QM, PW be drawn to the ſides ED, EF, adjoining the ſides to which the other perpendiculars were drawn, and let theſe perpendiculars be continued till MO be equal to QM and WS to PW; and proceed in the ſame manner with the remaining ſides if there be any.

Let the line SO which joins the extremities of the perpendiculars laſt drawn meet the ſides DE, EF to which theſe perpendiculars are drawn in D′ and E′, and draw D′Q meeting DC in C′ and E′P meeting AF in F′ and join BC′, BF′; then the angles ABF′, CBC′, DC′D′, ED′E′, FE′F, &c. to AF′B will be reſpectively equal to the angles a, b, c, d, e, f. That is, if the ſides bc, cd, de, ef and af be drawn parallel to the lines BC′, C′D′, D′E′, E′F, FB reſpectively, and ab be joined, the ſides of the polygon $abcdef$, &c. will make equal angles with the correſponding ſides of the given polygon.

Join CQ, AP, and let de produced meet CQ in R; alſo let ef *produced* meet AP in T. Then becauſe BL is equal to LQ and the angles at L right angles, the triangles $BC'L$, CLQ are equal in every reſpect, therefore the angle $cC'B$ is equal to the angle $cC'Q$, and conſequently the angle bcC, or CcR equal to the angle Dcd. In the ſame way it is ſhewn that the angle Ddc is equal to the angle Ede, the angle Eed to the angle FeS, the angle Ffe to the angle Afa, &c. Again it is evident that the right-angled triangles BCL, LCQ are equal in every reſpect; therefore BC is equal to CQ and the angle BCL to the angle QCL; and conſequently the triangles bcC, cCR having the angles at C and c equal, will alſo have the ſides bC, CR equal, therefore $BC - bc$ will be equal to $CQ - CR$, or Bb will be equal to RQ; and in the ſame manner it is ſhewn that aB is equal to PT. Now let OV be drawn to make the angle QOV equal to the angle CQO, and SV to make the angle VSP equal to the angle APS, and let de be produced to meet OV in N, SV in S, and SY, drawn parallel to OV in Y. Then, the triangles $D'QM$, $D'OM$, being in every reſpect equal, $D'Q$ will be equal $D'O$ and the angle $D'QM'$ to the angle $D'OM$; therefore, the angle RQD' is equal to the angle NOD', and the angles at D' and d being equal it is evident that the trapezoids $dRQD'$, $dNOD'$ are equal in every reſpect; therefore dR will be equal to dN and RQ or Bb to NO, that is, (becauſe of the parallels,) to SY. It is proved in the ſame manner that PT or aB is equal to SX.

On OV take OZ equal to CQ, and on SV take SU equal to PA, and join DZ, FU; then it is eaſily proved that the angle EDZ is equal to D, the angle DZV to $n - c$; the angle UFE to F and the angle FUV to $n - A$: alſo that the ſum of all the angles in the figure $VZDEFUV$ is equal to $4n$; therefore

$$ZVU + D + n - c + 2n - E + F + n - A = 4n,$$

or $ZVU + D + F = A + C + E = B + D + F$ (by hypotheſis); therefore ZVU, or its equal the angle YSX is equal to B; and conſequently the triangles YSX, aBb, having two ſides and the included angle equal, will be equal in every reſpect, therefore the angle $abB = SYX = dNV = dRc = Cbc$, and the angle $baB = SXY = cTA = Aaf$, therefore the ſides of the inſcribed polygon $abcdef$, make equal angles with the correſponding ſides of the given polygon as required.

It is evident that O and S are given points, and therefore the conſtruction for determining the angles is ſufficiently obvious.

Moreover, ſince be is equal to CR, and dN to dR, dN is equal to $dc + cb$; and ſince af is equal to fT and eX to cT, eX is equal to $ef + af$; and YX has been ſhewn to be equal to ab; therefore the perimeter of the inſcribed polygon is equal to NG, that is, becauſe of the parallels, to the given line OS.

If any other polygon as $a'b'c'd'e'f'$, &c. be inſcribed in the given one, ſo that the ſides $b'c'$, $c'd'$, $d'e'$, $e'f'$, $f'a'$ may be reſpectively parallel to the ſides bc, cd, de, ef, fa; then it is manifeſt from what has been done above, that the remaining ſides $a'b'$, ab, will alſo be parallel, and that the perimeter of the latter inſcribed polygon will alſo be equal to OS, or equal to the perimeter of the other inſcribed polygon.

Whence it appears that an indefinite number of polygons may be inſcribed in the given polygon, having their ſides reſpectively parallel and their perimeters equal to the ſame conſtant quantity, and yet leſs than the perimeter of any other polygon that can be inſcribed in the given one, and whoſe ſides make unequal angles with the corresponding ſides of the given polygon, which is a very curious and remarkable propoſition.

In the caſe of the rectangle it is obvious that the ſides of the inſcribed figure will be parallel to the diagonals, and the perimeter equal to the ſum of the two diagonals.

When the ſides of the inſcribed polygon cannot make equal angles with the correſponding ſides of the given polygon as in the caſes ſpecified in the ſolution, it appears that the problem is wholly indeterminate, and that the perimeter of the inſcribed polygon will continue to increaſe or decreaſe till one of the ſides vaniſhes or till two of the angular points of the inſcribed polygon coincide with one of the angular points of the given polygon.

MR. LOWRY *is requeſted to ſend to* MR. GLENDINNING's *for the Medal for ſolving the Prize Queſtion.*

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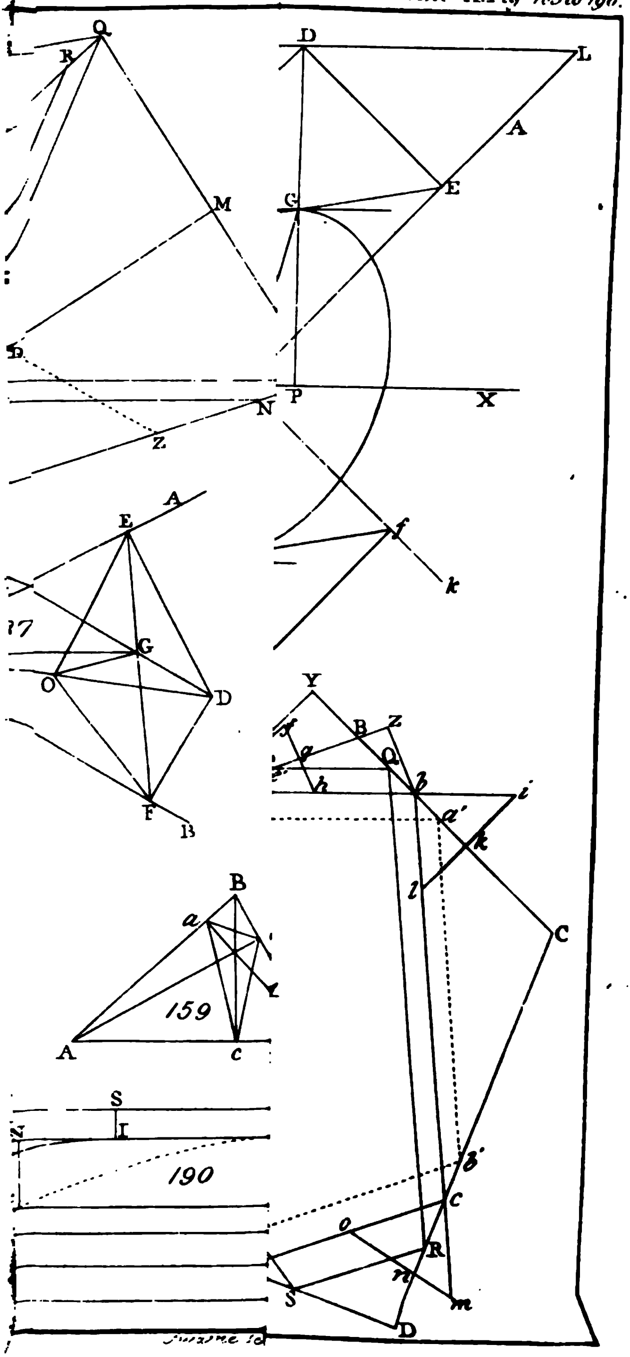
Plate X. Fig 183 to 196.



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NEW SERIES

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ORIGINAL ESSAYS.

ARTICLE I.

DEMONSTRATION *of ſome Propoſitions relating to ſuch portions of the Surface and Solidity of a Sphere as may be exactly Squared and Cubed.*

By Mr. JAMES IVORY, *of the Royal Military College.*

THE Problem propoſed by Viviani, a Mathematician of Florence, to the Geometers of his time, is well known, and has been often reſolved. Beſides effecting what was required by Viviani, the conſtruction given of the Florentine Problem leads to a number of curious properties; and, in addition to the properties that were known before, Boſſut, in the ſecond vol. of Memoirs of the National Inſtitute, announces a New Theorem, reſulting from the ſame conſtruction, and determining a portion of the ſolidity of a ſphere that may be exactly cubed, as the problem of Viviani determines a portion of the ſurface of a ſphere that may be exactly ſquared.

The demonſtrations that follow I believe to be new; and they are ſufficiently ſimple.

1. PROB. "If a circle be deſcribed upon any radius of the baſe of an hemiſphere, and upon the circle an upright cylinder be raiſed and be prolonged to pierce the hemiſphere; it is required to find the ſurface of the oval hole, made in the ſpherical vault."

Aſſume any point in the curve formed by the interſection of the ſurfaces of the ſphere and cylinder, and let x, y, z, be three co-ordinates determining the poſition of the aſſumed point; x, being perpendicular to the baſe of the hemiſphere; y, perpendicular to the diameter of the baſe of the cylinder; and z, the part of that diameter intercepted between y and the centre of the hemiſphere. Then if r denotes the radius of the hemiſphere, we ſhall have $r^2 = x^2 + y^2 + z^2$. And becauſe y is the ordinate, and z the abſciſs of a ſemicircle of which the diameter is r, $y^2 = (r - z) \times z = rz - z^2$. From the latter equation we get $y^2 + z^2 = rz$, which being ſubſtituted in the former, there reſults $r^2 = x^2 + rz$.

Let a great circle paſs through the aſſumed point and the pole of the hemiſphere; and let φ denote the arch of it intercepted between the aſſumed point and the baſe of the hemiſphere: then $x = r \text{ ſin. } \varphi$; $rz = r^2 - x^2 = r^2 \text{ coſ.}^2 \varphi$; therefore, $z = r \text{ cos.}^2 \varphi$; and $y^2 = rz - z^2 = r^2 \text{ coſ.}^2 \varphi - r^2 \text{ coſ.}^4 \varphi = r^2 \text{ coſ.}^2 \varphi \text{ ſin.}^2 \varphi$; therefore, $y = r \text{ coſ. } \varphi \text{ ſin. } \varphi$.

Let a plane parallel to the baſe of the hemiſphere be drawn through the aſſumed point; and let ρ denote the radius of the ſmall circle which is the common ſection of the plane and ſphere; then $\rho = r \text{ coſ. } \varphi$. Conceive alſo the plane of a great circle to paſs by the axis of the cylinder and the pole of the hemiſphere; it is obvious that this great circle will biſect the arch of the ſmall circle lying within the oval curve on the ſurface of the ſphere; it is plain too that y will be the ſine of half the arch of the ſmall circle to the radius ρ; therefore, the ſine of the like arch, in the circle having unity for radius, is $\frac{y}{\rho} = \frac{r \text{ ſin. } \varphi \text{ coſ. } \varphi}{r \text{ coſ. } \varphi} = \text{ſin. } \varphi$. Conſequently the meaſure of the half arch itſelf is φ; and the meaſure of the whole arch is 2φ. Therefore, the length of the arch of the ſmall circle, lying within the oval curve on the ſurface of the ſphere is

$$2\varphi \times \rho = 2r\varphi \text{ coſ. } \varphi.$$

But the diſtance of the ſmall circle from the baſe of the hemiſphere, reckoned on a great circle paſſing through the pole, is $r \times \varphi$; therefore, the fluxion of the part of the oval hole bounded by the arch of the ſmall circle is $r\dot{\varphi} \times 2r\varphi \text{ coſ. } \varphi = 2r^2\varphi\dot{\varphi} \text{ coſ. } \varphi$: Conſequently the part of the hole itſelf $= 2r^2 \times \int \dot{\varphi}\varphi \text{ coſ. } \varphi = 2r^2 \times \left(\varphi \text{ sin. } \varphi + \text{coſ. } \varphi\right) + C$; to determine C we have $2r^2 + C = 0$, whence $C = -2r^2$: And the corrected fluent is $2r^2 \times \left(\varphi \text{ sin. } \varphi + \text{coſ. } \varphi\right) - 2r^2$ which expreſſes the ſurface of the part of the hole lying between the ſmall circle and the baſe of the hemiſphere.

Let $\pi = 3{\cdot}1416$ &c. = ſemi-periphery to radius unity; then putting $\varphi = \frac{\pi}{2}$ in the above expreſſion, we ſhall find the whole ſurface of the oval hole made in the ſpherical vault $= r^2\pi - 2r^2$.

2. **Theor.** "If upon two radii that compoſe a diameter of the baſe of an hemiſphere two circles be deſcribed; and upon the two circles two upright cylinders be raiſed and be prolonged to pierce the hemiſphere; the portion of the ſurface of the hemiſphere that remains after the two oval holes are taken away will be equal to four times the ſquare of the radius of the hemiſphere."

For, according to what we have juſt found, the ſurfaces of the two oval holes $= 2r^2\pi - 4r^2$; but the whole ſurface of the hemiſphere $= 2r^2\pi$; therefore what remains, after the two holes are taken away, $= 2r^2\pi - \left\{ 2r^2\pi - 4r^2 \right\} = 4r^2$.

This theorem coincides with the reſult of the problem propoſed by Viviani. Let the two cylinders be prolonged, below the plane of their baſes, to pierce the other hemiſphere; and let the plane of a great circle paſs by the axes of the two cylinders. Then this great circle will divide the ſphere into two hemiſpheres, and will biſect the four oval curves formed on the ſurface of the ſphere; the four half-ovals that make part of the ſurface of each hemiſphere are the four windows of Viviani; and the remaining part of the ſurface of the hemiſphere is equal to $4r^2$ according to what has juſt been demonſtrated.

3. **Theor.** "The whole ſurface of the cylinder (1) included within the hemiſphere is equal to $2r^2$."

Montucla in giving the Hiſtory of the Florentine Problem, has noticed this property.

The fluxion of the baſe of the cylinder is $= \sqrt{[\dot{z}^2 + \dot{y}^2]}$ $\left\{ \text{becauſe } \dot{y} = \gamma\dot{\varphi}\ (\text{coſ.}^2\,\varphi - \text{ſin.}^2\,\varphi) \text{ and } \dot{z} = -2r\dot{\varphi} \times \text{sin.}\,\varphi\ \text{coſ.}\,\varphi \right\} = r\dot{\varphi}\sqrt{[(\text{coſ.}^2\,\varphi - \text{sin.}^2\,\varphi)^2 + 4\,\text{sin.}^2\,\varphi\ \text{coſ.}^2\,\varphi]} = \gamma\dot{\varphi}\sqrt{(\text{coſ.}^2\,\varphi + \text{sin.}^2\,\varphi)^2} = r\dot{\varphi}$.

But the fluxion of the ſurface of the cylinder bounded by $x = x\sqrt{[\dot{z}^2 + \dot{y}^2]} = r\dot{\varphi} \times x = r^2\dot{\varphi}\,\text{sin.}\,\varphi$; therefore, the ſurface

of

of the cylinder bounded by x, $= \int r^2\dot{\varphi}\,\text{sin.}\,\varphi = C - r^2\,\text{coſ.}\,\varphi$; to determine C we have $0 = C - r^2$ and $C = r^2$; therefore, the corrected fluent is $r^2 - r^2\,\text{coſ.}\,\varphi$; which expreſſes the ſurface of the cylinder bounded by x and inſiſting on the arch of the baſe lying between x and the point in which the baſe of the cylinder touches the baſe of the hemiſphere.

Put $\varphi = \frac{\pi}{2}$; and the fluent becomes ſimply r^2, which is equal to the ſurface of the cylinder inſiſting on half the circumference of the baſe; therefore, the whole ſurface $= 2r^2$.

4. THEOR. "If two cones be formed, having their vertices in the centre of the ſphere, and their baſes the two oval curves (2) formed on the ſurface of the ſphere; what remains of the whole ſolidity of the hemiſphere, after the two cones are taken away, will be equal to $\frac{4}{3}r^3$."

For, ſince the ſpherical ſurface which is the baſe of either cone has been ſhewn to be $= r^2\pi - 2r^2$, the ſolidity of the cone $= \frac{1}{3}r \times \left(r^2\pi - 2r^2\right) = \frac{r^3\pi}{3} - \frac{2r^3}{3}$. Therefore, the two cones $= \frac{2r^3\pi}{3} - \frac{4r^3}{3}$. But the whole ſolidity of the hemiſphere $= \frac{2r^3\pi}{3}$. Therefore, deducting the two cones, the remainder $= \frac{4r^3}{3}$.

5. THEOR. "If from the whole portion of the cylinder (1) included within the hemiſphere, the cone, that makes a part of it, be taken away, what remains will be equal to $\frac{2}{9}r^3$."

For let a plane be drawn through the centre of the ſphere to paſs by the ordinate x, and let s denote the ſolidity of the part bounded by this plane, the ſurface of the cylinder and the conical ſurface; from the point in which x meets the baſe of the hemiſphere, let a tangent be drawn to the baſe of the cylinder; and let p denote the perpendicular drawn to the tangent from the centre of the ſphere. Then $\dot{s}$ is *manifeſtly* equal to a pyramid of which p is the altitude and the fluxion of the ſurface of the cylinder $(= x\sqrt{[\dot{z}^2 + \dot{y}^2]})$ the baſe: therefore, $\dot{s} = \frac{1}{3}px\sqrt{[\dot{z} + \dot{y}^2]}$.

But the double of the triangle having its vertex in the centre of the ſphere, and $\sqrt{[\dot{z}^2 + \dot{y}^2]}$ for its baſe, is equal to $p\sqrt{[\dot{z}^2 + \dot{y}^2]}$. And the double of the ſame triangle is alſo equal to $(z\dot{y} - y\dot{z})$. Conſequently $p\sqrt{[\dot{z}^2 + \dot{y}^2]} = (z\dot{y} - y\dot{z})$.

Therefore, $\dot{s} = \frac{1}{3} \times x\,(z\dot{y} - y\dot{z}) = \frac{r^3}{3}\,\text{ſin.}\,\varphi\dot{\varphi} \times \left(\text{coſ.}^4\,\varphi - \text{ſin.}^2\,\varphi\ \text{coſ.}^2\,\varphi + 2\,\text{ſin.}^2\,\varphi\ \text{coſ.}^2\,\varphi\right) = \frac{r^3}{3}\,\text{ſin.}\,\varphi.\,\dot{\varphi} \times \left(\text{coſ.}^4\,\varphi + \text{coſ.}^2\varphi\,\text{ſin.}^2\,\varphi\right) = \frac{r^3}{3} \times \sin\varphi\,.\,\dot{\varphi} \times \text{coſ.}^2\,\varphi$.

Therefore, $s = C - \frac{r^3}{9}\,\text{coſ.}^3\,\varphi$: to determine C, we have $0 = C - \frac{r^3}{9}$ and $C = \frac{r^3}{9}$: And $s = \frac{r^3}{9} - \frac{r^3}{9}\,\text{coſ.}^3\,\varphi$; where s denotes the ſolid bounded by the ſurfaces of the cone and cylinder, and inſiſting upon the part of the area of the baſe of the cylinder lying between the diameter and the chord drawn from the centre of the ſphere through the point where x meets the baſe of the hemiſphere.

When $\varphi = \frac{\pi}{2}$, then $s = \frac{r^3}{9}$, which is the part of the ſolid inſiſting upon a ſemicircle of the baſe of the cylinder; therefore, the whole remaining part of the portion of the cylinder, where the cone is taken away, is equal to $\frac{4}{9}r^3$.

6. **Theor.** "If from the whole ſolidity of the hemiſphere, the portions of the two cylinders (2) included within it be taken away, what remains will be equal to $\frac{1}{9} \times (2r)^3$."

For we have already ſhewn (4) that the remainder of the hemiſphere, after the two cones are taken away, is equal to $\frac{4}{3}r^3$; and if we further take away the ſum of the exceſſes of the two cylindrical portions above the two cones (equal (5) to $\frac{4}{9}r^3$) the remainder will be $\frac{4}{3}r^3 - \frac{4}{9}r^3 = \frac{8}{9}r^3 = \frac{1}{9} \times (2r)^3$.

If the two cylinders be prolonged below the plane of their baſes to pierce the other hemiſphere, it is obvious that the portion taken away from the whole ſolidity of the ſphere by the two cylinders, leaves a remainder equal to $\frac{8}{9} \times (2r)^3$; which is preciſely the new theorem of Boſſut.

7. Let w denote the length of the arch of the curve of double curvature formed on the ſurface of the ſphere;

then $\dot{w}\sqrt{[\dot{y}^2+\dot{z}^2+\dot{x}^2]} = r\dot{\varphi}\sqrt{\Big((\text{coſ.}^2\,\varphi - \text{ſin.}^2\,\varphi)^2 + 4\,\text{ſin.}^2\varphi\,\text{coſ.}^2\varphi + \text{coſ.}^2\varphi\Big)} = r\dot{\varphi}\sqrt{\Big((\text{coſ.}^2\varphi + \text{ſin.}^2\varphi)^2 + \text{coſ.}^2\varphi\Big)} = r\dot{\varphi}\sqrt{[1+\text{coſ.}^2\,\varphi]} = r\dot{\varphi}\sqrt{[2-\text{ſin.}^2\varphi]} = \frac{r}{\sqrt{[2]}} \times \dot{\varphi}\sqrt{[1 - \tfrac{1}{2}\,\text{ſin.}^2\,\varphi]}$: Therefore, $w = \frac{r}{\sqrt{[2]}} \times \int \dot{\varphi}\sqrt{[1 - \tfrac{1}{2}\,\text{ſin.}^2\,\varphi]}$.

Now if an ellipſe be taken, of which the two axes are 1 and $\frac{1}{\sqrt{[2]}}$: then $\int \dot{\varphi}\sqrt{[1 - \tfrac{1}{2}\,\text{ſin.}^2\,\varphi]}$ will denote the length of an arch of the ellipſe, reckoned from the extremity of the conjugate axis, of which the ordinate, parallel to the tranſverſe axis, is ſin. φ. Therefore, the rectification of the curve of double curvature depends upon the rectification of an ellipſe having 1 and $\frac{1}{\sqrt{2}}$ for its axes.

ARTICLE II.

DEMONSTRATION *of a Theorem reſpecting Prime Numbers.*

By Mr. JAMES IVORY *of the Royal Military College.*

The following theorem, one of the moſt important in the theory of numbers, is the diſcovery of *Fermat*. It was publiſhed by him without demonſtration, as indeed were many other of thoſe curious properties of numbers that form the moſt difficult and intereſting part of the Diophantine Analyſis. The theorem was firſt demonſtrated by Euler in the Peterſburgh Commentaries; and his demonſtration may be found in a work on the Theory of Numbers, publiſhed by Le Gendre *.

I believe the following demonſtration to be new, and I reckon it more ſimple than that of Euler.

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* *Eſſai ſur la* Théorie des Nombres, Part 2de, Theor. 1re.

THEOREM.

THEOREM.

Let p be any prime number, and N any number not diviſible by p, then $N^{p-1} - 1$ will be diviſible by p.

DEMONSTRATION.

Let there be taken all the multiples of N, viz.

$$N,\ 2N,\ 3N,\ 4N\ \ldots\ldots\ldots\ldots\ (p-1)N$$

by the numbers in the natural ſeries 1, 2, 3, 4, &c. continued as far as $(p-1)$, or ſo long as the numbers in the ſeries are leſs than p; and let us conſider what will be the remainders if the ſeveral multiples be divided by p. That none of the multiples are diviſible by p is very evident: for if $m \times N$ be one of them, p does not divide N by hypotheſis, and m is leſs than p; therefore, p cannot divide the multiple $m \times N$. I likewiſe ſay, that the ſeveral remainders are all different from one another. For if it be poſſible let $m \times N$ and $n \times N$ be two multiples that, when divided by p, have the ſame remainder r; then

$$m \times N - r$$
$$n \times N - r$$

will be multiples of p: therefore, the difference $m \times N - n \times N = (m-n) \times N$ will likewiſe be a multiple of p: therefore, either $(m-n)$ or N muſt be a multiple of p, which is impoſſible; becauſe $m - n$ is leſs than p, and N is not a multiple of p by hypotheſis; therefore, no two of the multiples, when divided by p, can have the ſame remainder. But, from the nature of diviſion, the remainders muſt be all integer numbers leſs than p, and as they are juſt as many as the integer numbers leſs than p, and have been ſhewn to be all different from one another; it follows, that the ſeveral remainders can be no other than the ſeveral integer numbers leſs than p, or than the ſeveral numbers in the ſeries

$$1,\ 2,\ 3,\ 4\ \ldots\ldots\ldots\ldots\ldots\ (p-1).$$

Let us now adopt the notation $m{\cdot}p$ to denote a multiple of p in general, without attaching the notion of any particular number to the ſymbol m: then, becauſe the dividend is a multiple of the diviſor increaſed by the remainder, it is manifeſt from what has been proved that the multiples

$$N,\ 2N,\ 3N,\ 4N\ \ldots\ldots\ldots\ldots\ (p-1)N$$

may be repreſented by

$m{\cdot}p+1,\ m{\cdot}p+2,\ m{\cdot}p+3,\ m{\cdot}p+4,\ \ldots\ldots\ldots\ (m{\cdot}p+'p-1))$; each number in the one ſeries being equal to ſome one in the other ſeries; therefore,

$$N \times 2N \times 3N \times 4N \ldots\ldots\ldots\ldots \times (p-1)\,N =$$
$$(m{\cdot}p+1) \times (m{\cdot}p+2) \times (m{\cdot}p+3) \ldots\ldots\ldots\ldots \times (m{\cdot}p+(p-1)).$$

Now, if the expreſſion on the right-hand ſide of the equation be evolved, it will conſiſt of a certain number of terms, all of which, excepting the laſt, will contain p, or ſome power of p, as a factor; and the laſt term will be the continued product of all the numbers leſs than p: let the laſt term be denoted by Q, then it is clear that the right-hand ſide of the equation may be repreſented by $m{\cdot}p + Q$: It is alſo obvious that the left-hand ſide is equal to $Q \times N^{p-1}$: therefore $Q \times N^{p-1} = m{\cdot}p+Q$. Whence $Q \times N^{p-1} - Q = Q \times \{N^{p-1} - 1\} = m{\cdot}p$. Therefore, p neceſſarily divides either Q or $N^{p-1} - 1$: But Q can have no diviſors excepting numbers leſs than p or compoſite numbers formed by the product of numbers leſs than p multiplied together: conſequently p cannot divide Q; therefore it neceſſarily divides $N^{p-1} - 1$. *Q. E. D.*

Cor. Hence if N be any number not diviſible by the prime number p, the remainder of N^p when divided by p will be equal to the remainder of N divided by p.

For it has been ſhewn that $N^{p-1} - 1$ is always diviſible by p: therefore, when N^p and N are divided by p, the remainders of the diviſions muſt be the ſame.

ARTICLE

ARTICLE III.

FAGNANI'S *Theorem reſpecting Elliptic Arches, rendered more general. By Mr.* JAMES IVORY, *R. M. College.*

THE curious theorem of Count *Fagnani*, which aſſigns the difference of two Arches reckoned from the extremities of the quadrant of an ellipſe, is well known to Mathematicians: I propoſe at preſent to inveſtigate a more general theorem, which comprehends that of *Fagnani* as a particular caſe, and which aſſigns the difference of two elliptic arches reckoned from any two given points in the periphery of an ellipſe; and to deduce a ſimilar property in all the conic ſections.

1. Let ABC (fig. 96, pl. 5.) be a quadrant of an ellipſe, of which the ſemi-axis major is 1, and the excentricity ε: let the elliptic arch BD be denoted by w and the ordinate DE being drawn, put $CE = x$: then we ſhall readily obtain $\dot{w} = \dot{x}\sqrt{\left(\frac{1-\varepsilon^2 x^2}{1-x^2}\right)}$; or, putting $R = \sqrt{(1-\varepsilon^2 x^2)}$, $\dot{w} = \frac{R\dot{x}}{\sqrt{(1-x^2)}}$.

In like manner if the arch $Bd = w'$; $Ce = x'$; and $R' = \sqrt{(1-\varepsilon^2 x'^2)}$: then $\dot{w}' = \frac{R'\dot{x}'}{\sqrt{(1-x'^2)}}$.

Now $\frac{R^2-R'^2}{x'^2-x^2} = \varepsilon^2$: and if we put $y = x'\sqrt{(1-x^2)} + x\sqrt{(1-x'^2)}$, and $z = x'\sqrt{(1-x^2)} - x\sqrt{(1-x'^2)}$; then $y \times z = x'^2 - x^2$: conſequently $\frac{R^2-R'^2}{y \times z} = \frac{R+R'}{y} \times \frac{R-R'}{z} = \varepsilon^2$.

Suppoſe that the variable elliptic arch BD or ω becomes BM $(= m)$ when the variable elliptic arch Bd or w' becomes BN $(= n)$; and put p and q to denote CP & CQ, which are the correſponding values of CE & Ce or of x & x': then as the formula laſt inveſtigated is true, whatever x & x' denote it will be true, when p and q are ſubſtituted for x & x'; therefore, attending to what R, R′, y & z were put to denote, and if we further put

$$\left.\begin{aligned} a &= \frac{\sqrt{(1-\varepsilon^2 p^2)} + \sqrt{(1-\varepsilon^2 q^2)}}{q\sqrt{(1-p^2)} + p\sqrt{(1-q^2)}} \\ b &= \frac{\sqrt{(1-\varepsilon^2 p^2)} - \sqrt{(1-\varepsilon^2 q^2)}}{q\sqrt{(1-p^2)} - p\sqrt{(1-q^2)}} \end{aligned}\right\} \text{.............No. 1.}$$

it will eaſily appear that we ſhall have $a \times b = c^2$.

Therefore, $\frac{R + R'}{y} \times \frac{R - R'}{z} = a \times b$: and as each ſide of this equation is identically equal to c^2, it eſtabliſhes no relation between the variable quantities x & x', but leaves them quite independent. We are therefore at liberty to determine one of the two variable quantities x & x', by means of the other according to any relation that may beſt ſuit our purpoſe. Let us then ſuppoſe $R + R' = ay$; then on account of the preceding formula we ſhall neceſſarily have $R - R' = bz$. Hence by ſubſtituting for y and z, their values in x & x', we get

$$\left.\begin{array}{l} 2R = (a + b)\, x' \surd(1 - x^2) + (a - b)\, x \surd(1 - x'^2) \\ 2R = (a - b)\, x' \surd(1 - x^2) + (a + b)\, x \surd(1 - x'^2) \end{array}\right\} \text{No. 2.}$$

The equations juſt obtained, being the one a neceſſary conſequence of the other, only involve a certain relation between the two variable quantities x & x'; and I ſhall now endeavour to expreſs this relation in a formula as ſimple as poſſible. For this purpoſe, let the equations, No. 2, be ſquared and added together: then $2R^2 + 2R'^2 = (a^2 + b^2). \left\{x^2 (1 - x'^2) + x'^2 (1 - x^2)\right\} + 2(a^2 - b^2) \times x x' \surd \left\{(1 - x^2)(1 - x'^2)\right\} = 4 - 2c^2 (x^2 + x'^2) = 4 - 2ab\,(x^2 + x'^2)$.

Hence, $4 = (a + b)^2 \times \left\{x'^2 (1 - x^2) + x^2 (1 - x'^2)\right\} - (a - b)^2 x^2 x'^2 + 2(a^2 - b^2) \times x x' \surd\left\{(1 - x^2)(1 - x'^2)\right\}$.

Subtract both ſides from $(a + b^2$ and take the ſquare root; then

$$(a + b)\surd\left\{(1 - x^2)(1 - x'^2)\right\} - (a - b)xx' = \surd\left\{(a + b)^2 - 4\right\} \text{No. 3.}$$

which appears to be the ſimpleſt formula expreſſing the required relation of the variable quantities x & x'.

Multiply the firſt of the equations, No. 2, by $\frac{\dot{x}}{\surd(1 - x^2)}$ and the ſecond by $\frac{\dot{x}'}{\surd(1 - x'^2)}$: then, having ſubſtituted $\dot{w}$ for $\frac{R\dot{x}}{\surd(1 - x^2)}$ and $\dot{w}'$ for $\frac{R'\dot{x}'}{\surd(1 - x'^2)}$) and added the two equations together, there will reſult

$$\dot{w} + \dot{w}' = \frac{a + b}{2}(x'\dot{x} + x\dot{x}') + \frac{a - b}{2} \times \left\{\frac{x\dot{x}\surd(1 - x'^2)}{\surd(1 - x^2)} + \frac{x'\dot{x}'\surd(1 - x^2)}{\surd(1 - x'^2)}\right\}.$$

and taking the fluents

$$w+w'=\frac{a+b}{2}\times x'x-\frac{a-b}{2}\cdot\sqrt{\left\{(1-x^2)(1-x'^2)\right\}}+C,$$

an expreſſion that may be rendered more ſimple by exterminating the radical on the right-hand ſide by means of the formula, No. 3; it will then become

$$w+w'=\frac{2ab.\,xx'}{a+b}-\tfrac{1}{2}\cdot\frac{a-b}{a+b}\sqrt{\left\{(a+b)^2-4\right\}}+C.$$

To determine C, we have only to remark that when $w=m$ $w'=n$ and $p=x$ and $q=x'$: therefore,

$$m+n=\frac{2ab}{a+b}\cdot pq-\tfrac{1}{2}\cdot\frac{a-b}{a+b}\sqrt{\left\{(a+b)^2-4\right\}}+C:$$

conſequently, writing ε^2 for $a\times b$, we have

$$(w-m)-(n-w')=\frac{2}{a+b}\times\varepsilon^2\,(xx'-pq)\,\ldots\ldots\ldots\text{No. 4.}$$

that is, the difference of the elliptic arches MD and N*d*, reckoned from the two given points M and N, is equal to the algebraic quantity $\frac{2}{a+b}\times\varepsilon^2\,(xx'-pq)$: where a and b are to be determined from p and q by the formula, No. 1; and the relation of x and x' is expreſſed by the formula, No. 3.

If we ſuppoſe M to coincide with B, and N with A, (fig. 97, pl. 5,) then $p=0$ $q=\text{CA}=1$; and the formulas, No. 1, give $a=1+\sqrt{(1-\varepsilon^2)}$, $b=1-\sqrt{(1-\varepsilon^2)}$; and the formula, No. 3, gives $\sqrt{(1-\varepsilon^2)}=\sqrt{\left\{\frac{(1-x^2)(1-x'^2)}{xx'}\right\}}$: and in this caſe the difference of the two elliptic arches BD & A*d*, reckoned from the extremities of the quandrant, is equal to $\varepsilon^2\times xx'$: and this will be found to be nothing more than Fagnani's well known Theorem.

If x be taken equal to x'; ſo that the points D & *d* come together in H: then $\sqrt{(1-\varepsilon^2)}=\frac{1-x^2}{x^2}$, therefore $x^2=\text{CK}^2=\frac{1}{1+\sqrt{(1-\varepsilon^2)}}$; and the difference of the arches BH & HA $=\varepsilon^2x^2=\frac{\varepsilon^2}{1+\sqrt{(1-\varepsilon^2)}}=1-\sqrt{(1-\varepsilon^2)}=$ the difference of the semi-axis of the ellipſe.

2d. Let us now proceed to the hyperbola. Let C (fig. 98, pl. 5.) be the centre, and CA half the greater axis of an hyperbola:

Let w & w' denote the variable arches AD, Ad, and x & x' the diſtances CE & Ce of the correſponding ordinates from the centre: then if the ſemi-axis AC be 1, & the excentricity or diſtance of the focus from the centre be ε; we ſhall have $\dot{w} = \frac{\dot{x}\sqrt{(\varepsilon^2 x^2 - 1)}}{\sqrt{(x^2 - 1)}}$. and $\dot{w}' = \frac{\dot{x}'\sqrt{(\varepsilon^2 x'^2 - 1)}}{\sqrt{(x'^2 - 1)}}$: or writing R & R$'$ for $\sqrt{(\varepsilon^2 x^2 - 1)}$ and $\sqrt{(\varepsilon^2 x'^2 - 1)}$; $\dot{w} = \frac{R\dot{x}}{\sqrt{(x^2-1)}}$ and $\dot{w}' = \frac{R'\dot{x}'}{\sqrt{(x'^2-1)}}$

It is manifeſt that $\frac{R^2 - R'^2}{x^2 - x'^2} = \varepsilon^2$: and if we put $y = x'\sqrt{(x^2-1)} + x\sqrt{(x'^2 - 1)}$ and $z = x'\sqrt{(x^2 - 1)} - x\sqrt{(x'^2 - 1)}$ we ſhall likewiſe have $y \times z = x^2 - x'^2$: therefore $\frac{R + R'}{y} \times \frac{R - R'}{z} = \varepsilon^2$.

Suppoſe that when the point D coincides with M, that d coincides with N: put AM $= m$ and AN $= n$ and let p and q denote CP and CQ which are the contemporary values of x and x': Let

$$\left.\begin{aligned} a &= \frac{\sqrt{(\varepsilon^2 p^2 - 1)} + \sqrt{(\varepsilon^2 q^2 - 1)}}{q\sqrt{(p^2 - 1)} + p\sqrt{(q^2 - 1)}} \\ b &= \frac{\sqrt{(\varepsilon^2 p^2 - 1)} - \sqrt{(\varepsilon^2 q^2 - 1)}}{q\sqrt{(p^2 - 1)} - p\sqrt{(q^2 - 1)}} \end{aligned}\right\} \ldots\ldots\ldots\ldots \text{No. 1.}$$

then $a \times b = \varepsilon^2$.

Therefore, $\frac{R + R'}{y} \times \frac{R - R'}{z} = a \times b$; a formula which holds good whatever x, x', p and q, denote. Now, by purſuing the reaſoning exactly in the ſame manner as has been done in the caſe of the ellipſe, we ſhall find

$$\left.\begin{aligned} 2\,R &= (a+b)\,x'\sqrt{(x^2 - 1)} + (a-b)\,x\sqrt{(x'^2 - 1)} \\ 2\,R' &= (a-b)\,x'\sqrt{(x^2 - 1)} + (a+b)\,x\sqrt{(x'^2 - 1)} \end{aligned}\right\} \ldots\text{No. 2,}$$

where x and x' have to one another the relation expreſſed in the formula.

$$(a+b)\times\sqrt{\left\{(x^2-1)\,(x'^2-1)\right\}} + (a-b)xx' = \sqrt{\left\{(a+b)^2 - 4\right\}} \quad \text{No. 3.}$$

Let the firſt of the equations, No. 2, be multiplied by $\frac{\dot{x}}{\sqrt{(x^2-1)}}$ and the ſecond by $\frac{\dot{x}'}{\sqrt{(x'^2-1)}}$; then having added the reſults together,

gether, we ſhall find an equation that may be integrated, and of which the fluent is

$$w+w' = \frac{a+l}{2}\,xx' + \frac{a-b}{2}\sqrt{\left\{(x^2-1)(x'^2-1)\right\}} + C \,..\,\text{No. 4.}$$

from this equation the quantity C will be exterminated by means of m, n, p and q, which are contemporary values of w, w', x and x': and if, for the ſake of ſimplicity, we further exterminate the radical quantity by means of the formula, No. 3, we ſhall finally obtain

$$\left(w-m\right) - \left(n-w'\right) = \frac{2}{a+b} \times \varepsilon^{2}\left(\, xx' - pq \,\right) ...\text{No. 5.}$$

that is, the difference of the hyperbolic arches MD and Nd, reckoned from the given points M and N, and correſponding to x and x' of which the valuation is expreſſed by the formula, No. 3, is equal the algebraic quantity $\frac{2}{a+b} \times \varepsilon \times (xx' - pq)$.

The property that we have juſt inveſtigated for the hyperbola is equally ſimple with that before found for the ellipſe, and is quite analogous to it. We have thus another property, in addition to ſo many others already known, common to the two curves, and admitting of an enunciation in the ſame words. Many curious properties may be deduced from the expreſſions, marked No. 4 and No. 5: but I ſhall here confine my attention to one only, which effects for the hyperbola what *Fagnani's* theorem does for the ellipſe. The property I allude to is, in effect, contained in the formula, No. 4, and may be derived from it by making p and q equal to their extreme limits; that is, by taking p infinitely great and $q = 1$: but, as this method of deduction would introduce notions that, in ſtrictneſs, may be objected to, I ſhall give a rigid inveſtigation of it.

Reſume the formula $\frac{R+R'}{y} \times \frac{R-R'}{z} = \varepsilon^{2}$ which ſuppoſes no relation between x and x'; and let us ſuppoſe that theſe variable quantities are ſo related that $R + R' = \varepsilon y$; then, of conſequence, $R - R' = \varepsilon z$; whence, by reſtoring the values of y and z:

$$R = \varepsilon' x \surd\; x^2 - 1)$$
$$R' = \varepsilon x \surd(x'^2 - 1).$$

And we ſhall readily find that the relation of x and x', which theſe formula involve, may be thus ſimply expreſt;

$$\varepsilon\sqrt{\left\{(x^2-1)\;(x'^2-1)\right\}} = \sqrt{(\varepsilon^2-1)} \,..............\, \text{No. 6.}$$

Multiply the firſt of the two formulas by $\frac{\dot{x}}{\surd(x^2 - 1)}$, and the ſecond by $\frac{\dot{x}'}{\surd(x'^2 - 1)}$; add the reſults together; take the fluents; and there will come out

$$w + w' + C = \varepsilon xx' \ldots\ldots\ldots \text{No. 7.}$$

Draw the aſymptote of the hyperbola, (fig. 99, pl. 5.) and produce the ordinates DE, *de* to meet it in S and R: draw AF parallel to the ordinates: take CT a fourth proportional to CF ($=\varepsilon$), CR and CS, and the formula, No. 7, amounts to this, "That the exceſs of the line CT, above the ſum of the two hyperbolic arches A*d*, AD, is a conſtant quantity C, provided x and x' ſatisfy the formula, No. 6."

The formula, No. 6, ſhews that x becomes indefinitely great when x' approaches indefinitely to unity: whence it is eaſy to infer that the conſtant quantity C is the exceſs of the whole aſymptote infinitely produced above the whole hyperbolic curve infinitely continued.

The conſtant quantity C will be moſt ſimply determined from the caſe when $x = x'$: in this caſe the formula, No. 6, gives $x^2 = \frac{\varepsilon + \surd(\varepsilon^2 - 1)}{\varepsilon}$ whence this conſtruction. "Take FG = FA, and CL a mean proportional between CF and CG: then, the conſtant quantity C, or the exceſs of the hyperbolic curve above the aſymptote, both infinitely continued, will be equal to the exceſs of the line CG, above twice the hyperbolic arch AH."

I add one more theorem. "Having determined the points H and G as above, let $de \times \text{DE} = \text{HK}^2$; and take CT a fourth proportional CF, CR and CS: then the exceſs of the arch HD above the arch H*d* will be equal to the ſtraight line GT: or, which is the ſame thing, the exceſs of the ſum of the arches AD and A*d*, above twice the arch AH, will be equal to the ſtraight line GT."

By this Theorem the rectification of an Arch AD greater than AH, is reduced to the rectification of another arch A*d* leſs than AH.

34. We proceed now to the parabola. Let AE (fig. 100, pl. 5.) be the axis: put AE $= x$, A*e* $= x'$ and let w and w' denote the correſponding arches of the parabola: then, 4ε denoting the parameter of the axis, we have, $\dot{w} = \frac{\dot{x}}{\surd(x)} \surd(\varepsilon + x)$ and $\dot{w}' = \frac{\dot{x}'}{\surd(x')} \surd(\varepsilon + x^2)$; or, writing R and R′ for the radicals

$\surd(\varepsilon + x)$ and $\surd(\varepsilon + x')$, $\dot{w} = \frac{R\dot{x}}{\surd(x)}$, and $\dot{w}' = \frac{R\dot{x}}{\surd x'}$.

It is manifeſt that $\frac{R^2 - R'^2}{x - x'} = \frac{R + R'}{\surd x + \surd x'} \times \frac{R - R'}{\surd x - \surd x'} = 1$, whatever x and x' denote. Suppoſe AM $(= m)$ and AN $(= n)$ to be contemporary values of w, & w' and AP $= p$ and AQ $= q$ to be the correſponding values of x and x': put

$$\left.\begin{array}{l} a = \frac{\surd(\varepsilon + p) + \surd(\varepsilon + q)}{\surd(p) + \surd(q)} \\ b = \frac{\surd(\varepsilon + p) - \surd(\varepsilon + q)}{\surd(p) - \surd(q)} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \text{No. 1.}$$

then $a \times b = 1$: and $\frac{R + R'}{\surd x + \surd x'} + \frac{R - R'}{\surd x - \surd x'} = a \times b$.

The formula juſt found ſuppoſes no relation between the variable quantities x and x': we may, therefore, ſuppoſe $R + R' = a\surd x + a\surd x'$; and it will follow that $R - R' = b\surd x - b\surd x'$: whence

$$2R = (a + b)\surd x + (a - b)\surd x'$$
$$2R' = (a - b)\surd x + (a + b)\surd x'$$

two formulas that involve a relation of x and x', more ſimply expreſſed by the equation

$$4\varepsilon = (a - b) \times \left\{ a(\surd x + \surd x')^2 - b(\surd x - \surd x')^2 \right\} \text{No. 2}$$

Multiply the two formulas for R and R' by $\frac{\dot{x}}{\surd x}$ and $\frac{\dot{x}'}{\surd x'}$ reſpectively, and having added them together we ſhall ſtill find an equation that is integrable, and of which the fluent is

$$w + w' = C + \frac{a + b}{2} \times \left(x + x'\right) + (a - b) \times \surd(xx') \ldots \text{No. 3.}$$

and, if we further exterminate the radical $\surd(xx')$ by means of the formula, No. 2, and change the conſtant quantity we ſhall find

$$w + w' = \frac{2}{a + b} \times \left(x + x'\right) + C \ldots\ldots\ldots\ldots\ldots \text{No. 3.}$$

In this formula C will be determined by any contemporary values of w, w', x and x', as m, n, p and q: thus we have

$$(w - m) - (n - w') = \frac{2}{a + b} \times \left\{ (x - p) - (q - x') \right\} \text{No. 4.}$$

That is the difference of the parabolic arches MD and ND is to the difference of PE and eQ as $(a + b)$ to 2.

I ſhall not proſecute this ſubject any further at preſent, but may reſume it on another occaſion.

ARTICLE

ARTICLE IV.

A GEOMETRICAL PORISM, *with two Examples of its application to the Solution of Problems.* By SCOTICUS.

THE following Poriſm is extremely ſimple and obvious; it affords, however, a good example of the peculiar nature of thoſe propoſitions, and of their uſe in the ſolution of geometrical problems.

PORISM. Fig. 101. Pl. 5.

Let A and B be two given points, and CDE a circle given by poſition, there is a given point P in the ſtraight line joining the points A, B, ſuch, that if from P any ſtraight line whatever be drawn, meeting the circle in C and D, the points A, B, C, D, ſhall be in the circumference of a circle.

Suppoſe the poriſm to be true; take C′ any given point in the circumference of the given circle, and join C′P meeting the circle again in D′. Becauſe the lines PDC, PD′C′ meet the circle, $CP \cdot PD = C'P \cdot P'D'$; but by hypotheſis the points A, B, C, D, are in the circumference of a circle, and, therefore, $CP \cdot PD = AP \cdot PB$, therefore, $AP \cdot PB = C'P \cdot PD'$, ſo that the points A, B, C′, D′, are alſo in the circumference of a circle; now A, B, C′ are given points, therefore, the circle paſſing through them and D′ is given by poſition, and hence the point D′ and the line C′D′ are given by poſition; therefore P, the point in which it meets AB, is given, as was to be ſhewn.

CONSTRUCTION. Take any point C′ in the circumference of the given circle, and deſcribe a circle through the points A, B, C′, meeting the given circle again in D′; join C′D′, meeting AB in P, the point required; that is, if from P, any ſtraight line be drawn, meeting the circle in C and D, the points A, B, C, D, will be in the circumference of a circle. The ſynthetic demonſtration is ſufficiently evident, as alſo the limitation of the data.

PROBLEM I.

(Simpſon's Geometry, Problem 45,) Fig. 101. Plate 5.

To deſcribe a circle that ſhall paſs through two given points *A, B,* and touch another circle C, E, D, given by poſition.

Construction. Take any point C′ in the circumference of the given circle, and deſcribe a circle through the given points A, B, C′, which will touch the given circle in a point D, (in which caſe the thing required is done,) or will cut it again in a point D′; join C′ D′ meeting AB in P, and draw PE touching the given circle in E; a circle deſcribed through A, B, E, is evidently that which is required; the demonſtration is evident.

PROBLEM II.

(Simpſon's Geometry, Prob. 15, or Select Exerciſes, Prob. 49.) Fig. 102. Plate 5.

From two given points A, B, to draw two ſtraight lines AD, BD to meet in a ſtraight line MN, given by poſition, ſo that $AD + BD$, may be equal to a given line.

Construction. On B, one of the given points as a centre, with a radius equal to $AD + BD$, the given line, deſcribe a circle; from A, the other point, draw AH perpendicular to MN, and produce the perpendicular ſo that $HK = HA$: through the points A, K, deſcribe any circle whatever that may meet the other circle in F and G; join GF meeting AK in C, and draw CE, alſo Cε touching the circle whoſe centre is B in E and ε; join BE, Bε, meeting MN in D and δ, and draw AD, Aδ; then AD, BD, alſo Aδ, Bδ, are the lines required.

Demonstration. Becauſe $KC \cdot CA = GC \cdot CF = EC^2$, the line EC is a tangent to a circle paſſing through A, K, E, therefore, its centre is in the line EB; but becauſe MN biſects AB at right angles, its centre is alſo in MN, therefore, it is at D the interſection of EB and MN; hence $AD = DE$, and $AD + DB = EB$, the given line. In the ſame way it may be ſhewn that $A\delta + \delta B =$ the given line.

* * *

ARTICLE V.

GEOMETRICAL PROPOSITIONS.

By Mr. Ivory, *R. M. College.*

THE Propoſitions of which I propoſe to give the Geometrical Analyſis, were publiſhed without Solutions by Dr. Stewart, at the end of his General Theorems: and they were given by him as ſome of the properties of the circle that had occurred in the inveſtigation of thoſe remarkable propoſitions. They are of that claſs called by the ancients, Poriſms; the nature o which, before very obſcure, has been fully elucidated by the learned Profeſſor Playfair, in a memoir that completes the reſtoration of the moſt difficult branch of the ancient Geometry.

PROPOSITION I. PORISM. Fig. 103. Pl. 5.

Let there be a circle given by poſition, and let A, B, be two given points: a point C may be found within the circle, ſuch, that if through the point C there be drawn any line meeting the circle in D and E, and AD, BD, AE, BE, be joined; the rectangle ADB will be to the rectangle AEB as CD to CE.

ANALYSIS.

Take O, the centre of the circle, and join OB: draw GH through C at right angles to BO, and join AG, GB, AH, HB: join alſo AC, and let it meet BO in F. Then, becauſe GH is drawn through C, by hypotheſis,

$$AG \times GB : AH \times HB :: GC : CH;$$

but, becauſe GH is at right angles to BO paſſing through the centre of the circle, it will be biſected by BO: therefore, BG = BH, as it is eaſy to prove: therefore,

$$AG : AH :: GC : CH.$$

Therefore, the line AC will biſect the angle GAH. Suppoſe a circle to be deſcribed about the triangle GAH: then, becauſe the angles GAF, HAF are equal, AF will paſs through the middle point of the arch of the circle ſubtended by GH: and becauſe BO biſects GH at right angles, B will paſs through the ſame point: therefore, F, cannot but be the middle point of that arch. Therefore, $AC \times CF = GC \times CH = DC \times CE$.

From what has been ſhewn, it follows that the points A, E, F, D, are in the circumference of a circle: let the circle be deſcribed, and let FK be a diameter: draw DP, EQ, perpendicular to AF. Then, (Prop. C. VI. Simſon's Euclid) $AD \times DF = FK \times DP$ and $AE \times EF = FK \times EQ$:

therefore,

$$AD \times DF : AE \times EF :: DP : EQ :: DC : CE.$$

But, by hopotheſis,

$$AD \times DB : AE \times EB :: DC : CE;$$

therefore,

$$AD \times DF : AE \times EF :: AD \times DB : AE \times EB.$$

Hence, it is eaſy to infer, that
FD : is DB : : FE : EB.

Attending now to what is demonſtrated in the analyſis of the ſecond Propoſition of the ſecond Book of the *Loci Plani* of Apollonius, we muſt conclude from the laſt analogy, that the rectangle BO × OF, is equal to the ſquare of the ſemidiameter of the given circle. Therefore, the point F is given; and the line AF is given by poſition. Since then A and F are given points, and ſince what has been ſhewn in regard to DE is true of any chord paſſing through C, we may aſſume the point D any where in the circumference of the given circle: then the circle ADE will be determined, and of conſequence the point E: therefore the point C, which is the interſection of AF and DE, will be given.

Hence this conſtruction: draw BO to the centre of the given circle; take OF a third proportional to BO, and the ſemidiameter, and join AF; aſſume any point in the circumference of the given circle (not in the ſame ſtraight line that the points A and F are), and through A, D, F, deſcribe a circle cutting the given circle again in E: join DE, cutting AF in C, and C will be the point ſought.

The curious property of the point inveſtigated in this Propoſition may be made more general, as in the following Theorem. In order to render the demonſtration more ſimple a lemma is premiſed.

PROPOSITION II. LEMMA. Fig. 104. Pl. 5.

In the triangle ACB, let CD biſect the angle ACB, and let AE and AF be drawn to make equal angles (BAE, CAF) with the ſides AB and AC, and to meet CD in E and F: then BE and BF being drawn, AE will be to EB as AF to FB.

From F and E draw the perpendiculars FG, FH, FN, EK, EL, and EM: then becauſe CD biſects the angle ACB, it is obvious that FG = FH and EK = EL. Becauſe the angle BAE is equal to CAF; therefore the triangle MAE will be ſimilar to the triangle FAG; and the triangle NAF to EAK: therefore,

AE : EM :: AF : FG
AE : EK :: AF : FN.

Conſequently;

EM : EK :: FG : FN;

that is,

EM : EL :: FH : FN.

Now the angles MEL, and NFH, are likewiſe equal to one

another,

another, for each of them is the ſupplement of the angle at B: therefore the triangle MEL is ſimilar to HFN, and, conſequently, the angle HNF is equal to the angle MLE. But the angle HNF is equal to HBF, and the angle MLE to MBE, as is obvious: therefore the angle HBF is equal to MBE, and the triangles FBH and EBM are ſimilar to one another;

Therefore,

$$EB : EM :: FB : FH;$$

but as was before ſhewn,

$$AE : EM :: AF : FG \text{ or } FH.$$

Therefore,

$$AE : EB :: AF : FB \ldots\ldots\ldots\ldots \textit{Q. E. D.}$$

Cor. If AE and AF be drawn to make equal angles with BA and AC. Then BE and BF will make equal angles with BA and BC.

PROPOSITION III. THEOREM. Fig. 105. Pl. 5.

Let there be a circle, of which O is the centre, and two points, A and B, as in Prop. 1.; and let C be the point found in that Propoſition: draw AO, OB, to the centre, and through C draw MN parallel to OT, the line that biſects the angle AOB: let CD, CE, be any how drawn from C to meet the circle in D and E, and to make equal angles with MN, and join DA, DB, EA, EB, then the rectangle ADB is to the rectangle AEB as CD to CE.

DEMONSTRATION.

Let RS be drawn to biſect AF at right angles, and let it meet OB in S: draw OH at right angles to DC, and OG at right angles to EC, and let OH and OG meet RS in H and G; join AH, AG, AS, DF and EF: draw OR perpendicular to RS, and DP, EQ to AF: and let DC, and EC, meet the circle again in *d*, *e*.

Then, becauſe DC, EC, make equal angles with MN, therefore OH and OG, which are perpendicular to DC, CE, will likewiſe make equal angles with MN, or with OT parallel to MN: and becauſe the angles AOT, BOT, are equal, as well as the angles HOT, GOT, therefore the angle AOH is equal to the angle BOG. Again, becauſe RS biſects AF at right angles, it is eaſy to infer that AS will be equal to SF, and that the angle ASO will be biſected: therefore applying to the triangle AOS what was demonſtrated in the preceding lemma, it will be

$$OH : HA :: OG : GA;$$

whence it is eaſy to ſhew, that

$$OH \times DP : OG \times EQ :: 2\,HA \times DP : 2\,GA \times EQ.$$

From the nature of the point C, the four points A, D, F, *d*, are in the circumference of a circle: now the centre of that circle muſt be in the line OH, which is perpendicular to D*d*, and biſects D*d*; and the ſame centre muſt likewiſe be in the line RS, becauſe RS biſects AF at right angles: therefore, H the concurrence of OH and RS, muſt be the centre of the circle paſſing through the point A, D, F, *d*. Therefore, 2HA is equal to a diameter of the circle paſſing through A, D, F, *d*; and the rectangle $2HA \times DP$ is equal to $AD \times DF$ (Prop. C. VI. Simſon's Euclid). In like manner it is ſhewn, that $2\,GA \times EQ = AE \times EF$. Therefore, on account of the analogy before demonſtrated, $OH \times DP : OG \times EQ :: AD \times DF : AE \times EF$.

Becauſe OH is perpendicular to DC, and RS to AF, therefore the angle DCA is equal to the angle OHR, and the triangle DCP is ſimilar to OHR. Therefore $OH : OR :: CD : DP$, and, conſequently, $OH \times DP = OR \times CD$. In like manner it is ſhewn that $OG \times EQ = OR \times EC$.

Therefore,

$$OH \times DP : OG \times EQ :: OR \times CD : OR \times CE :: CD : CE.$$

And combining this, with what is ſhewn above,

$$AD \times DF : AE \times EF :: CD : CE.$$

Becauſe, $BO \times OF$, is equal to the ſquare of the ſemidiameter of the circle, it follows, ſomewhat is demonſtrated in the ſecond Propoſition of the ſecond Book of the *Loci Plani* of Apollonius, that

$$FD : DB :: FE : EB.$$

Hence,

$$AD \times DF : AE \times EF :: AD \times DB : AE \times EB.$$

But, as has been ſhewn,

$$AD \times DF : AE \times EF :: CD : CE;$$

therefore,

$$AD \times DB : AE \times EB :: CD : CE.$$

Q. E. D

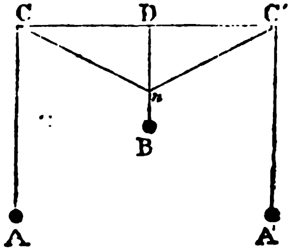
(To be continued.)

ARTICLE VI.

Solution of a Dynamical Question, by Mr. IVORY.

THE following queſtion in Dynamics is Prop. 3d. page 131, of Simpſon's Miſcellaneous Tracts; and it is alſo reſolved in Mr. Atwood's Treatiſe on Rectilineal Motion: but the ſolutions of theſe two authors bring out very different reſults. Mr. George Sanderſon, of London, having pointed out the above circumſtance to Mr. Leybourn (in a letter, an extract from which was ſhewn to me), I was led to conſider the problem; and, in conſequence, found out the ſolution I am now to explain. The queſtion is thus propoſed by Simpſon:

"Suppoſe that a thread, ACnC′A′, having two equal weights, A,A′, ſuſpended at the ends thereof, is hung over two tacks, C,C′, in the ſame horizontal line; and that to the middle point of the thread (n), equally diſtant from the tacks, another given weight B is fixed, which is permitted to deſcend by its own gravity, ſo as to cauſe the other two weights, at the ſame time, to aſcend; it is propoſed to find the law of the velocity by which the ſaid weights aſcend and deſcend, abſtracting from the reſiſtance of the air the weight of the thread, and the friction on the tacks."



SOLUTION.

Draw nD perpendicular to the horizontal line CC′; then it is obvious that CD = C′D. Let CD = a, Dn = x, and Cn = y: alſo let $g = 32\frac{1}{6}$, the meaſure of the accelerating force of gravity. Put $\dot{\tau}$ to denote an indefinitely ſmall interval, or the fluxion of the

the time, which is ſuppoſed to flow uniformly: then the velocity of the weight A in aſcent $= \frac{\dot{y}}{\dot{\tau}}$: and the variation of the velocity $= \frac{\ddot{y}}{\dot{\tau}^{2}}$. But the weight A cannot aſcend at all, unleſs its gravity be counteracted, and an additional force be impreſſed upon it equivalent to produce the actual velocity in aſcent. Therefore, the force (conſtantly acting on each particle of A), neceſſary to cauſe A to aſcend with the velocity $\frac{\dot{y}}{\dot{\tau}}$ is meaſured by $\left(g + \frac{y^{2}}{\dot{\tau}}\right)$ and the whole motive force acting on A is, therefore, $= \left(g + \frac{\ddot{y}}{\dot{\tau}^{2}}\right) \times$ A, an expreſſion that, for the ſake of brevity, we ſhall denote by F. Now the motive force F, by which A is urged upward, being derived from the action of the weight B; therefore B will be reacted upon in the direction nC by an equal motive force, F. But the force in the direction nC may be reſolved into two others: one $= \frac{CD}{Cn} \times F = \frac{a}{y} \times F$, urging B in a direction parallel to DC; and another force $= \frac{nD}{nC} \times F = \frac{x}{y} \times F$, urging B in the direction nD, and retarding the deſcent of B.

Becauſe the weight A is equal to the weight A′, and that each of them acts in a ſimilar manner on the weight B: therefore, the effect of the weight A′, is, in like manner, reſolved into two forces acting on B; one in the direction DC′, and preciſely equal to the former force in the direction DC, and, therefore, deſtroying its effect: and another force in the direction nD, likewiſe equal to the former force in that direction, but conſpiring with it to retard the deſcent of B. Therefore, the whole effect of the two weights A and A′ on the weight B, is equivalent to a motive force $= 2 \times \frac{x}{y} \times F = \frac{x}{y} \times \left(g + \frac{\ddot{y}}{\dot{\tau}^{2}}\right) \times 2A$, directly oppoſing the deſcent of B.

Again, the velocity with which B actually deſcends is $= \frac{\dot{x}}{\dot{\tau}}$ and the variation of that velocity, or the accelerating force with which B actually deſcends $= \frac{\ddot{x}}{\dot{\tau}}$. Therefore the accelerating force conſtantly deſtroyed by the connection of B with the weight A, $A' = g - \frac{\ddot{x}}{\dot{\tau}^2}$: and the whole motive force conſtantly loſt $= \left(g - \frac{\ddot{x}}{\dot{\tau}^2}\right) \times B$.

Now as the motive force conſtantly loſt by B, muſt be equal to the motive force conſtantly retarding its deſcent, we ſhall have,

$$\frac{x}{y} \times \left(g + \frac{\ddot{y}}{\dot{\tau}^2}\right) \times 2A = \left(g - \frac{\ddot{x}}{\dot{\tau}^2}\right) \times B$$

where $y^2 = a^2 + x^2$.

Now $y\dot{y} = x\dot{x}$: therefore $\frac{x}{y} = \frac{\dot{y}}{\dot{x}}$. Let $\frac{B}{2A} = m$; then by ſubſtitution and properly ordering the terms, we get,

$$\frac{m \times \ddot{x}\dot{x} + \ddot{y}\dot{y}.}{\dot{\tau}^2} = g\,(m\dot{x} - \dot{y})$$

and taking the fluents, recollecting that $\dot{\tau}$ is conſtant,

$$\frac{m\dot{x}^2 + \dot{y}^2}{2\dot{\tau}^2} = g \times (mx - y) + d$$

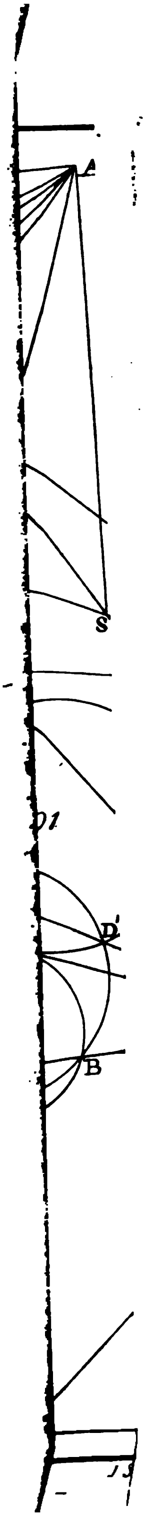
where d is a conſtant quantity neceſſary to correct the fluent.

In the laſt equation ſubſtitute $\frac{x^2\dot{x}^2}{y^2}$ for its equal $\dot{y}^2$, and alſo v for $\frac{\dot{x}}{\dot{\tau}}$, which is the meaſure of the velocity of B, correſponding to nD or x, and we ſhall obtain

$$v = y \times \frac{\sqrt{2g\,(mx - y) + 2d}}{\sqrt{mx^2 + y^2}}$$

which correſponds exactly with the expreſſion for the velocity obtained by Simpſon.

Therefore, by the ſolution juſt given, the accuracy of Simpſon's concluſion is confirmed.


A
S
D
B

ARTICLE VII.

Of the equilibrium of a very long and ſlender cylinder floating in a fluid.

By Jac. Rube.

1. LET BC (fig. 138, pl. 8,) repreſent the ſurface of a fluid, in which floats the long and ſlender cylinder AD, conſidered in the firſt place as a material line: and let it be propoſed to examine the caſe when the extremity A is reſiſted by a fixed obſtacle, as the edge of the veſſel containing the fluid. Let G be the centre of gravity of the whole cylinder, and g that of the part CD immerſed in the fluid; then if we conceive the part CD to be divided into a number of ſmall portions, each of theſe portions will be urged in a vertical direction, by a force equal to the weight of the fluid which it diſplaces. Therefore, the total effect of the action of the fluid on the cylinder will be equivalent to a ſingle force acting at g, in a vertical direction, and equal to the ſum of all the partial forces, that is, to the weight of a portion of the fluid equal in bulk to the part of the cylinder CD. Therefore the forces that act on the cylinder are theſe three; a force (q) acting at g in a vertical direction gF, equivalent to the weight of a portion of the fluid equal in bulk to the part CD; the force of gravity, equal to the weight of the whole cylinder (w), acting at G in a direction GE parallel, but oppoſite to gF: and thirdly, a force acting in the direction AN perpendicular to the axis of the cylinder, the effect of the reaction of the fixed obſtacle, and equal to the preſſure at A.

It is now clear that no equilibrium can take place in the ſuppoſed circumſtances. For if a ſolid body be in *equilibrio* by means of the action of three forces, it is eſſentially neceſſary that the directions of the three forces meet in a point, or that they be all three parallel to one another. Now the cylinder AD being inclined to the horizon, the direction of the force AN cannot be parallel to the directions of the other two forces; therefore an equilibrium cannot take place.

Take gF to repreſent the force q, and GE to repreſent the force of gravity w. Draw gM, GK perpendicular to the axis of the cylinder, and complete the parallelograms LM, HK: alſo let φ denote the angle ACB, or the inclination of the cylinder to the horizon. Then the force q is equivalent to a force $= q \times$ coſ. φ in the direction gM, and a force $= q \times$ ſin. φ in the direction gL; and in like manner, the force w is equivalent to a force $= w \times$ coſ. φ in the direction GK, and a force $= w \times$ ſin. φ in the direction GH.

The force gL oppoſes the force GH, and their difference $\left\{= (w-q)\ \text{ſin.}\ \varphi\right\}$ only acts on the cylinder, and is employed in urging it to ſlip off the ſupport A. No part of this force can be deſtroyed by the force AN acting at right angles to it. We may ſuppoſe it to be counter-balanced by a weight $\mathrm{P} = (w-q)$ ſin. φ acting in the direction of the axis of the cylinder by means of a thread paſſing over a pulley.

The remaining forces gM, GK, and AN, are now all parallel, and may be in equilibrio. The conditions are, firſt, that the ſum of the forces gM and AN be equal to the force GK; and, ſecondly, that the force gM be to the force AN as GA to Gg.

Thus we have

$$\mathrm{GA} : \mathrm{G}g :: q\ \text{coſ.}\ \varphi : (w-q)\ \text{ſin.}\ \varphi$$

whence

$$\mathrm{GA} : g\mathrm{A} :: q : w.$$

The weight $\mathrm{P} = (w-q)$ ſin. φ, becomes ſmaller and ſmaller as the angle φ becomes leſs and leſs: and the angle φ may be ſo very ſmall that the friction at A, together with the reſiſtance of the fluid, may be a ſufficient counterpoiſe to the tendency to ſlip. This muſt be ſuppoſed to be the caſe in the 10th queſtion of the Gentleman's Diary for 1789; for otherwiſe the equilibrium ſuppoſed in the queſtion could not ſubſiſt. Taking the numbers in that queſtion; AD $= 36$ inches, the diameter of the cylinder $= \frac{3}{4}$ of an inch, AC reckoned on the axis $= 13$ inches, AB $= 3$ inches and the weight of a cubic inch of the fluid $= 0{\cdot}5949$ oz. avoirdupois. Hence $q = \frac{9}{16} \times 0{\cdot}7854 \times 23 \times 0{\cdot}5949 = 6{\cdot}045$ oz. nearly; and A$g = 24\frac{1}{2}$; AG $= 18$: therefore $18 : 24\frac{1}{2} :: 6{\cdot}045 : 8{\cdot}228 = w =$ weight of the cylinder required.

The angle φ will be determined from this equation $3 = 13\ \text{ſin.}\ \varphi - \frac{3}{8}\ \text{coſ.}\ \varphi$; whence ſin. $\varphi = 0{\cdot}2585$ and coſ. $\varphi = 0{\cdot}9660$: therefore the force AN, or the preſſure at A $= (w-q)$ coſ. $\varphi = 2{\cdot}099$ oz. and the weight P, equivalent to the tendency to ſlip off the ſupport A, $= (w-q)$ ſin. $\varphi = 0{\cdot}564$ oz. or ſomething more than half an ounce.

2. If it be required to have regard to the dimenſions of the cylinder, in the ſolution of ſuch problems as the foregoing, it will become neceſſary to determine the centre of gravity of the part of a right cylinder cut off by a plane oblique to the axis.

For this purpoſe let it be required to determine the centre of gravity of the cylindrical hoof, ABC (fig. 139, pl. 8.) Let the oblique elliptical ſurface of the hoof touch the circular baſe in A, and let ACB be a ſection by a plane paſſing through A and the axis of the cylinder: alſo let DEGF be a ſection by a plane perpendicular both to the baſe and to ACB: then it is eaſy to prove that the ſection EF is a rectangle, that DH = HE and FK = KG, and that, having drawn KH, it is parallel to BC and = FD or GE. Biſect BC in L, and draw AL meeting KH in M: then, it is obvious, that KM = MH, and that the point M is the centre of gravity of the rectangular ſection EF. Thus, the centres of gravity of all the ſections of the hoof by planes parallel to EF are in the line AL: therefore the centre of gravity of the hoof itſelf will be in the line AL.

Let S repreſent the ſuperficial content of the ſection of a ſolid homogeneous body by a plane, and x the diſtance of the ſection from a given parallel plane; then $\int S x\dot{x}$ (taken between the limits of the greateſt and leaſt values of x, that is, from the value of x where S is neareſt the given plane to the value of x where x is moſt remote from that plane) = the ſolid content of the body multiplied by the diſtance of the centre of gravity from the given plane. To apply this formula to the caſe in hand, put φ = meaſure of the arch BD, BC $= h$, and r = radius of the circular baſe: then DE $= 2r\ \text{ſin.}\ \varphi$, DF $= \text{KH} = \frac{\text{BC}}{\text{BA}} \times \text{AH} = \frac{h}{2r} \times r\,(1 + \text{coſ.}\ \varphi) = \frac{h}{2}\,(1 + \text{coſ.}\ \varphi)$; therefore $S = hr\,(1 + \text{coſ.}\ \varphi)\ \text{ſin.}\ \varphi$: alſo $x = r\,(1 - \text{coſ.}\ \varphi)$: therefore $S x\dot{x} = hr^3 \times \dot{\varphi}\ \text{ſin.}^2\ \varphi\,(1 + \text{coſ.}\ \varphi)\,(1 - \text{coſ.}\ \varphi) = hr^3 \times \dot{\varphi}\ \text{ſin.}^4\ \varphi = hr^3 \times \dot{\varphi} \left\{ \frac{3}{8} - \tfrac{1}{2}\,\text{coſ.}\ 2\,\varphi + \frac{1}{8}\ \text{coſ.}\ 4\ \varphi \right\}$: therefore $\int S x\dot{x} = hr^3 \times \left\{ \frac{3}{8}\,\varphi - \tfrac{1}{4}\,\text{ſin.}\ 2\,\varphi + \frac{1}{32}\ \text{ſin.}\ 4\,\varphi \right\}$: and taking the whole fluent from $x = o$ to $x = 2r$, or from $\varphi = o$ to φ = ſemi-circumference $= \pi$, we have $\int S\,x\dot{x} = hr^3 \times \frac{3\pi}{8}$. Now the ſolidity of the hoof = a cylinder on the ſame baſe and half the height $= hr^2 \times \frac{\pi}{2}$. Therefore the diſtance of the centre of gravity of the hoof from a plane touching the cylindrical ſurface in BC $= \frac{hr^3 \times \frac{3}{8}\pi}{hr^2 \times \frac{1}{2}\pi} = \frac{3}{4}\,r$. Hence, if o be the centre of the baſe, take $om = \frac{1}{4}\,r$, and draw mn parallel to BC, meeting AL in n, and n will be the centre of gravity of the hoof required.

It is now eaſy to find the centre of gravity of a portion of a right cylinder cut off by a plane oblique to the axis. Let $PwoQ$ be the axis, r the middle of oQ, or the centre of gravity of the cylinder AN; join nr, and make nT to Tr, as the cylinder AN to the hoof ABC, that is, as OL to OP, and T will be the centre of gravity of the whole portion of the cylinder cut off by the oblique plane AGFC.

Draw ns and Tq perpendicular to PQ: Then $sw = \frac{1}{4}\, wo = \frac{1}{8}\, Po$ and $rw = \frac{1}{2}\, PQ$; therefore $rs = \frac{1}{2}\, PQ + \frac{1}{8}\, PO$: but $Po : QP :: rT : nr :: qr : rs$;

therefore $rq = \frac{1}{2}\, Po + \frac{1}{8} \cdot \frac{Po^2}{PQ}$. Hence

$$Qq = \frac{1}{2}\, PQ + \frac{1}{8} \cdot \frac{Po^2}{PQ}$$

$$Tq = \frac{1}{4} \cdot \frac{AO \cdot OP}{PQ}.$$

3. Now let it be required to reſolve the problem conſidered in § 1, taking into account the dimenſions of the cylinder. Let G (fig. 140. pl. 8.) be the middle of the axis AD, or the centre of gravity of the whole cylinder, and s the middle of the part CD: put $AD = a$, $As = b$, $2r =$ diameter of the cylinder, and $\varphi =$ angle ACB $=$ inclination of the cylinder to the ſurface of the fluid: then take $sr = \frac{r^2 \cot.^2 \varphi}{16\,(a-b)}$ and $rg = \frac{r^2 \cot. \varphi}{8\,(a-b)}$, and g will be the centre of gravity of the immerſed part of the cylinder (§ 2). Then proceeding as in § 1, the force of gravity GE ($= w =$ weight of the cylinder) will be reſolved into two forces, the one $GK = w \cos. \varphi$, acting at right angles to the axis of the cylinder; and the other $GH = w \sin. \varphi$, acting parallel to the axis: and in like manner, the force of the fluid on the immerſed part of the cylinder, gF ($= q =$ weight of a portion of fluid equal in bulk to the immerſed part of the cylinder) will be equivalent to two other forces, one $gM = q \cos. \varphi$, perpendicular to AD, and one $gL = q \sin. \varphi$, parallel to DA.

The forces GH and gL oppoſe one another: join Gg, and produce it p, making $w - q : q :: Gg : Gp$, and draw pn parallel to the axis of the cylinder: then the two oppoſing forces GH and gL will be equivalent to a ſingle force applied to p or n and $= (w - q) \sin. \varphi$, the difference of the forces GM and gL. No part of this force can be deſtroyed by the preſſure on the obſtacle (friction being excluded) which reacts on the cylinder in a direction a direction perpendicular to the axis. We may ſuppoſe it to be counter-balanced by a weight $P = (w - q)\ \text{ſin.}\ \varphi$, acting in the direction pn by means of a thread, faſtened to the point n, paſſing over a pulley. The diſtance of n from the centre of the cylinder, or $nA = \frac{r^2 \cot.\ \varphi}{8\,(a-b)} \times \frac{q}{w-q}$.

From the conditions neceſſary to the equilibrium of the other three forces, we eaſily deduce, as in § 1,

$$GA : rA :: q : w$$

or in ſymbols

$$\tfrac{1}{2}\ a : b - \frac{r^2 \cot.^2 \varphi}{16\,(a-b)} :: q : w$$

and the preſſure on the obſtacle $= (w - q)\ \text{coſ.}\ \varphi$.

Taking the numbers in § 1, we have $q = 6{\cdot}045$ oz: $a = 36$; $b = 24\tfrac{1}{2}$: and $\varphi = 14^\circ\ 59'$: and $r = \frac{3}{8}$: whence $18 : 24{\cdot}489 :: 6{\cdot}045 : 8{\cdot}224$ oz. $= w$ the weight of the cylinder required.

4. Suppoſe now that AD (fig. 141, pl. 8.) is a long cylindrical beam of timber, moveable at one end round an axis (paſſing through the centre of the cylinder at right angles to its axis) supported by a prop AB of a given height above the ſurface of a fluid in which the beam floats: it is required to determine the poſition in which the beam will reſt in equilibrio.

Let c = height of the centre of the axis, round which the beam turns, above the ſurface of the fluid; a = length of the cylinder, and φ = inclination to the ſurface of the fluid: then $\frac{c}{\text{ſin.}\ \varphi}$ = length of the part of the axis of the cylinder out of the fluid, and $(a - \frac{c}{\text{ſin.}\ \varphi})$ = length of the immerſed part: let w = weight of the cylinder, w' = weight of a portion of fluid equal in bulk to the cylinder, then $(1 - \frac{c}{a\ \text{ſin.}\ \varphi}) \times w'$ = weight of a portion of fluid equal in bulk to the immerſed part of the cylinder. Take $Dr = \tfrac{1}{2}\,(a - \frac{c}{\text{ſin.}\ \varphi}) + \frac{1}{8} \cdot \frac{r^2 \cot.^2 \varphi}{a - \frac{c}{\text{ſin.}\ \varphi}}$ ($2r$ being the diameter of the cylinder) and $g\,r = \frac{1}{4} \cdot \frac{r^2 \cot.^2 \varphi}{a - \frac{c}{\text{ſin.}\ \varphi}}$;

then will g be the centre of gravity of the immerſed part of the cylinder: draw the vertical line gF meeting the axis AD in m;

then $rm = gr \times \tan.\ \varphi = \frac{1}{4} \cdot \frac{r^2}{a - \frac{c}{\text{ſin.}\ \varphi}}$: hence Am = AD,

$$Dr - rm = \tfrac{1}{2}\left(a + \frac{c}{\text{ſin.}\ \varphi}\right) - \frac{1}{8} \cdot \frac{2r^2 + r^2 \cot.\ \varphi}{a - \frac{c}{\text{ſin.}\varphi}}.$$

The only forces that act on the beam are the force of gravity $= w$ (the weight of the beam) in the direction GE, and the force $gF = \left(1 - \frac{c}{a\ \text{ſin.}\ \varphi}\right) \times w'$, and the beam will be in equilibrio in that poſition where theſe two forces are in equilibrio, that is, when

AG : Am :: force gF : force GE

or, in ſymbols,

$$\frac{a}{2} : \tfrac{1}{2}\left(a + \frac{c}{\text{ſin.}\varphi}\right) - \frac{1}{8} \cdot \frac{2r^2 + r^2 \cot.^2 \varphi}{a - \frac{c}{\text{ſin.}\ \varphi}} :: \left(1 - \frac{c}{a\ \text{ſin.}\ \varphi}\right) w' : w.$$

$$\text{Hence } a^2 - \frac{c^2}{\text{ſin.}^2\ \varphi} - \frac{r^2}{4} \times (2 + \cot.^2\ \varphi) = a^2 \cdot \frac{w}{w'},$$

$$\text{and ſin.}\ \varphi = \sqrt{\frac{c^2 + \frac{r^2}{4}}{a^2\left(1 - \frac{w}{w'}\right) - \frac{r^2}{4}}}.$$

If in the formula juſt found, we make $c^2 + \frac{r^2}{4} = a^2\left(1 - \frac{w}{w'}\right) - \frac{r^2}{4}$ or $c = \sqrt{\left\{a^2\left(1 - \frac{w}{w'}\right) - \frac{r^2}{2}\right\}}$, then ſin. $\varphi = 1$ and $\varphi = 90°$: therefore when c has this value, and much more when c has a greater value (in which caſe the formula gives the value of ſin. φ greater than the radius) the beam will not reſt in equilibrio in any poſition oblique to the horizon: but, when c has a leſs value, there will be one poſition, oblique to the horizon, as determined by the formula, in which the beam will reſt in equilibrio.

Let it be remarked, however, that the formula is limited in its application; for it will not apply when the prop is ſo ſhort, that either the lower edge of the upper end of the cylinder is immerſed in the fluid, or the upper edge of the lower end of the cylinder *riſes out* of the fluid. To inveſtigate theſe caſes it would be neceſſary to determine the centre of gravity of a hoof of a right cylinder inſiſting on a given part, or ſegment of the baſe.

5. Suppoſe now that the long and ſlender cylinder (fig. 142, pl. 8.) AD, floating in a fluid, is kept in equilibrio by a weight P, acting on the cylinder by means of a thread ANP paſſing over a pulley, and faſtened to the extremity of the axis of the cylinder: and let it be required to determine the weight P when the axis of the cylinder has a given inclination to the ſurface of the fluid.

Although this problem may be eaſily deduced from the problem laſt conſidered, yet we ſhall give an independant ſolution of it.

Let G be the middle of the axis, or the centre of gravity of the cylinder; and S the middle of the immerſed part of the axis ↓; put $AD = a$, $SD = x$, and take $Dr = x + \frac{1}{8} \cdot \frac{r^2 \cot.^2 \varphi}{2x}$ (φ being the given inclination of the axis to the ſurface of the fluid, and $2r$ the diameter of the cylinder) and $rg = \frac{1}{4} \cdot \frac{r^2 \cot. \varphi}{2x}$: then g will be the centre of gravity of the immerſed part of the cylinder; draw the vertical line gF cutting the axis in m, then $rm = gr . \tan. \varphi = \frac{1}{4} \cdot \frac{r^2}{2x}$: therefore $Am = a - x - \frac{1}{8} \cdot \frac{r^2 \cot.^2 \varphi}{2x} - \frac{1}{4} \cdot \frac{r^2}{2x}$. Alſo if w = weight of the cylinder, and w' = weight of a portion of fluid equal in bulk to the cylinder, then $\frac{2x}{a} \times w'$ = weight of a portion of fluid equal in bulk to the immerſed part of the cylinder.

The forces that act on the cylinder are the two vertical forces, applied to the points m and A, equal to $\frac{2x}{a} \times w'$ and the weight P; and the force in the direction of gravity, applied to G, and equal to w.

The conditions of equilibrium are firſt that $w = \frac{2x}{a} \times w' + P$: and, ſecondly, that $AG : Gm :: \frac{2x}{a} \times w' : P$ or $w - \frac{2x}{a} w'$: whence $AG : Am :: \frac{2x}{a} \times w' : w$, and $AG \times w = Am \times \frac{2x}{a} \times w'$, or in ſymbols:

(a —

$$\left(a - x - \frac{1}{8}\cdot\frac{r^2 \cot.^2 \varphi}{2x} - \frac{1}{4}\cdot\frac{r^2}{2x}\right) \times \frac{2x}{a} \times w' = \frac{a}{2} \times w:$$

Thus $4ax - 4x^2 = a^2 . \frac{w}{w'} + \frac{1}{2} r^2 + \frac{1}{4} r^2 \cot.^2 \varphi$

therefore $a - 2x$, or $AC = \sqrt{\left\{a^2 \left(1 - \frac{w}{w'}\right) - \frac{1}{2} r^2 - \frac{1}{4} r^2 \cot.^2 \varphi\right\}}$:

and x being thus known, the value of $P = w - \frac{2x}{a} \times w'$ will likewiſe be known.

But it muſt be remarked, that this formula will not apply when the cylinder is ſo obliquely inclined to the ſurface of the fluid, that either the lower edge of the upper end of the cylinder is immerſed in the fluid, or the upper edge of the lower end riſes out of the fluid.

If in the formula that has been found we put $r = 0$, then $AC = a \sqrt{\left(1 - \frac{w}{w'}\right)}$, which agrees with the reſult of Mr. Farey's inveſtigation, (page 161, No. 14, of the Old Series of the Repoſitory).

Taking the numbers in the experiment made by Mr. Farey, and of which ſome account is given in the Repoſitory as above cited, we have $a = 28{\cdot}12$ inches, $2r = 1{\cdot}7$ inches, and $w = 17{\cdot}65$ ounces: the experimenter informs us, that when the cylinder was placed upright and allowed to ſink to its equilibrium in the water, he obſerved 13·92 inches of the axis to be immerſed; but that when the ſame thing was done, inverting the cylinder, it ſunk 14·45 inches into the water: the former of theſe trials gives $w' = \frac{28{\cdot}12}{13{\cdot}98} \times 17{\cdot}65$ oz. and the latter gives $w = \frac{28{\cdot}12}{14{\cdot}45} \times 17{\cdot}65$ oz.; and it appears reaſonable to prefer a mean of the two trials, and thus to take $w' = \frac{28{\cdot}12}{14{\cdot}215} \times 17{\cdot}65$ or $\frac{w}{w'} = \frac{14{\cdot}215}{28{\cdot}12}$. If theſe numbers be ſubſtituted in the formula we ſhall find $AC = \sqrt{(390{\cdot}648 - 0{\cdot}108 \cot.^2 \varphi)} =$ (by throwing the root into a ſeries, and rejecting all the terms but the two firſt on account of their ſmallneſs,) $19{\cdot}76 - 0{\cdot}0045 \times \cot.^2 \varphi$. The Author of the experiment informs us, that he obſerved 19·5 inches of the cylinder's axis to be above the water at ſeveral inclinations; and the theory, which gives 19·76 inches inſtead of *19·5 agrees* as well with the fact as can reaſonably be expected,

confidering

conſidering the uncertainty in the determination of w'. The correction depending on the inclination of the axis of the cylinder is manifeſtly too ſmall to be of any account in the preſent inſtance.

We have CD, or $2x$, $= 8{\cdot}36$ in $+ {\cdot}0045 \cot.^2 \varphi$: therefore $\frac{2x}{a} w' = \frac{17{\cdot}65}{14{\cdot}215} + \left\{8{\cdot}36 + {\cdot}0045 \cot.^2 \varphi\right\} = 10{\cdot}29$ oz. $+ {\cdot}0055 \cot.^2 \varphi$. In Mr. Farey's experiment the thread AN, faſtened to the end of the cylinder, was attached to one ſcale of a balance, the weight P being put into the oppoſite ſcale: and we are informed that, while the balance was raiſed and depreſſed by means of a pulley, ſo as to make the cylinder paſs into all degrees of obliquity to the ſurface of the fluid, the ſcales remained *as nearly as poſſible* in equilibrio, and the thread vertical, the counterpoiſe, or weight P, being $= 7{\cdot}175$ oz. neglecting the correction depending on the inclination, the theory gives 7·36 inſtead of 7·175: but when $\varphi = 6°$, then cot. $\varphi = 9\frac{1}{2}$ nearly, and the correction depending on the inclination = half an ounce nearly: thus according to theory the counterpoiſe, or weight P, ought to diminiſh from 17·36 oz. to 6·86 oz., while the cylinder paſſes from being perpendicular to the horizon to an inclination of 6°. The theory ſeems to agree with Mr. Farey's experiment as nearly as his obſervations afford us the means of judging. When the inclination of the axis, in this experiment, is between 5° and 6°, the upper edge of the lower end of the cylinder would, according to theory, juſt come to the ſurface of the water, and after this the formula we have given would no longer apply: but the great length to which this article has already run, renders it improper to add any thing more on this ſubject at preſent.

ARTICLE VIII.

Of the length of an arc of a circle in terms of the tangent.

By Mr. Benjamin Gompertz.

LET w and v be the arcs correſponding to the tangents z and t reſpectively, radius being unity.

Aſſume

$$w = az + bz^2 + cz^3 + dz^4 + \&c.$$

$$\text{and } v = at + bt^2 + ct^4 + dt^4 + \&c.$$

a, b, c, d, &c. being conſtant quantities to be determined.

PROPOSITION II. PORISM. Fig. 144, Pl. 8.

Two ſtraight lines, AC, CB, being given in poſition, and therein two points, A, B, there is *given* a point D, ſuch, that any circle thence deſcribed ſhall cut off ſegments AE, BF, adjacent to the given points, having between themſelves a *given* ratio.

Analysis. Suppoſe it true: and let D be the point which is affirmed to be *given*, and let the circle meet the lines again in K, L; then, by hypotheſis, the ratio of AK to BL is *given*, and the ſame with that of AE to BF, therefore the ratio of EAK to FBL is *given*, that is, the ratio of $AD^2 - DG^2$ to $BD^2 - DH^2$ is given; alſo DG = DH, and AD, DB, are *given*; now ſince the aforeſaid ratio is given it is either of equality or not;—if not, let which of the terms you will as $AD^2 - DG^2$ be the greater, then, dividendo $AD^2 - BD^2$ has a *given* ratio to $BD^2 - DH^2$, therefore ſince AD, BD, are *given*, DH is alſo *given*, contrary to the hypotheſis; therefore the ratio is of equality, and AD = BD; and therefore, ſince AE = BF and AK = BL, EK = FL, that is, they are equidiſtant from D; whence D is given, being the interſection of DO, which biſects and is perpendicular to AB, and CD which biſects the angle ACL; and it is evident that D may be found in either of the four angles at C.

PROPOSITION III. PORISM.

If there be any given number of points, and a circle given by poſition; putting *n* for the given number, if a polygon of $n+1$ ſides be inſcribed in the circle ſo that a ſide may paſs through each of the given points, the remaining ſide ſhall contain a *given* angle at the circumference with a ſtraight line tending to a *given* point.

Case 1. If in the circle CDE, given in poſition, the triangle DCE be inſcribed ſo that DC, CE, tend to A, B, given points, ED makes a *given* angle at D, with a line that tends to a *given* point. Fig. 145, pl. 8.

Analysis. Suppoſe it true, and let F be the point which in the poriſm is affirmed to be *given*, then if DF be drawn meeting the circle in G, the angle GDE is *given*; join AB, parallel to which draw EH and join DH, meeting AB in K, then the angle DCE = KHE = DKA, therefore KA × AB = DA × AC; therefore K is a given point. Join GH, then the angle *GHE = EDF* is *given*, but HE is parallel to a ſtraight line AB given in poſition; therefore HG is alſo parallel to a ſtraight line *given* in poſition; join KF meeting the circle in T,V; then, ſince K is given and F *given*, KV is *given* in poſition, parallel thereto draw HX, then ſince HG and HX are parallel to lines *given* in poſition, the angle XHG is *given*, therefore GX is *given* in magnitude: alſo, ſince KDF is inflected from the points F, K, and HX drawn parallel to FK, and GX joined meets FK in S, S may be ſhewn to be *given* in the ſame way that K was. Now, ſince through the *given* point S is drawn GX *given* in magnitude but not in poſition, S is the centre, and GX conſequently a diameter, and XHG a right angle, therefore the angle $FDE = GHE$ $=$ difference of $BKS = XHF$ and a right angle, but K and S are given, therefore the angle SKB is given, and therefore the angle FDE is given, and ſince $KF \times FS = TF \times FV$*, F is given, *viz*. by making $TK : KV :: TF : FV$ .... *Q. E. I.*

This is the 57th prop. of Dr. Simſon's treatiſe de poriſmatibus, and the laſt poriſm of the third book of Euclid's Poriſms.

It may not be improper to enumerate its particular caſes. The angle FDE, that is, the difference between the given angle SKB and a right angle may be either a right angle or nothing, that is, the angle SKB may be nothing (or SK coincide with AB, AB paſſing through the centre) or a right angle, in both caſes K is found by making $KA \times AB = DA \times AC$.

When AB paſſes through the centre the angle FDE is a right angle, and here we have two particular caſes.

Firſt, if K and F are the given points, HG is perpendicular to the diameter, i. e. HG makes a right angle with HX, which tends to a point at an infinite diſtance. This caſe is prop. 58, de poriſmatibus.

Secondly, which is in fact the converſe, if KHD be drawn and HG perpendicular to KS, then DG tends to a given point F: i. e. HG is drawn from a point infinitely diſtant in a line perpendicular to the diameter and paſſing through K; this is prop. 62, de poriſmatibus, caſe the fifth.

When AB does not paſs through the centre, there are likewiſe two particular caſes:

Firſt, when SK is perpendicular to AB then the angle EDF is nothing, that is, ED paſſes through the given point; this is prop. 61, de poriſmatibus.

Secondly, when AB is given in poſition and from K, any point in it is drawn KD, and HE tending to a point infinitely diſtant in it, i. e. parallel to AB, DE ſtill makes a given angle EDF, found as before, with a line tending to the given point F. This is the ſecond caſe of prop. 62, de poriſmatibus.

The analyſis of every one of theſe caſes is contained in the general analyſis, the general propoſition having been reduced to them one after another; but I have been thus preciſe in the enumeration of them that the beautiful connection between the propoſitions cited, might be clearly ſeen. I may alſo obſerve, that this analyſis is in every reſpect different from Dr. Simſon's, except that both are grounded upon the principle of the analyſis of prop. 117, lib. 7. Pappi, which Dr. Simſon has made, prop. 55, de poriſmatibus.

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* By the ſimilar triangles DFK, GFS.

LEMMA I. Fig. 146, Pl. 8.

If from a given point D, to a circle given in poſition, be any how drawn DC, and from C be drawn CA in a given angle DCA, and AB be drawn parallel to a ſtraight line given in poſition, and BC be joined; BC makes at C a *given* angle with a ſtraight line that tends to a *given* point.

Demonstration. Join BG, then the angle ABG is given, becauſe ACD is given, therefore BG, making a given angle with a line that is parallel to a line given in poſition, is itſelf parallel to a line given in poſition, therefore the lemma is evident by that caſe of the preceding poriſm that makes prop. 62, de poriſmatibus.

LEMMA II. Fig. 147, Pl. 8.

If from two given points D, E, the ſtraight lines DA, AE, be inflected to any point A in a given circumference meeting it again in C, F, and from A be drawn AB to the circumference in a given angle EAB, BC joined makes at C a *given* angle with a line tending to a *given* point.

Demonstration. Join FC, which, by the foregoing poriſm, makes a given angle with a line tending to a given point G, but the angle BCF (= EAB) is given, therefore the angle BCG is given.

Case 2. When there are three or more points F, G, H, K, &c.

Demonstration. Having drawn the diagonals AC, AD, &c fig. 148, plate 8. By the firſt caſe AC makes a *given* angle with a line that tends to a given point, therefore by the ſecond lemma AD alſo makes a given angle with a line tending to a given point, and, therefore, by the ſame lemma, AE alſo makes a given angle with a line tending to a given point; and ſo on, let the number of given points be what it may. Therefore the propoſition is evident.

Cor. Hence it is evident how a polygon may be inſcribed in a circle ſo that its ſides may paſs through any given number of given points, or be parallel (lem. 1.) to any given number of lines given by poſition, or ſome paſs through given points and ſome be parallel to, or even (lem. 2.) make given angles with lines tending to given points.

The following poriſm is added here from its ſimilarity to the firſt caſe of the foregoing, and as a further ſpecimen of my method of inveſtigating this claſs of propoſitions.

PROPOSITION IV. PORISM. Fig. 149, Pl. 8.

A circle whoſe centre is A, and a ſtraight line BC being given in poſition and magnitude, if through B, C, any circle be deſcribed cutting the given circle in D, E; DE joined tends to a *given* point.

Analysis. Suppoſe the poriſm true, and F the point *given*, join FC meeting the circle again in G, then CF × FG (= EF × FD) is *given*, becauſe F is *given* and the circle HED *given*, but FC is *given* in magnitude and poſition, therefore FG is *given* in magnitude and poſition; therefore G is a *given* point; therefore G coincides with B, for otherwiſe the circle CBG would be given in poſition, which it is not; join FA and biſect BC in K, then ſince it has been proved above that $(KF)^2 - (KC)^2 = (AF)^2 - (AH)^2$, the difference of the ſquares of KF and AF, being equal to the difference of the ſquares of KC and AH, is given; therefore (1 prop. 2, lib. loc. plan.) F touches a ſtraight line given in poſition, and it has been proved to be in BC produced if need be; therefore F is given .... *Q. E. I.*

And it appears from the propoſition cited, that if AK joined is perpendicular to BC, F is at an infinite diſtance, i. e. DE is parallel to BC.

ARTICLE X.

DIOPHANTINE PROBLEMS.

By Mr. James Cunliffe, *R. M. College.*

PROBLEM I.

TO find three ſquare numbers ſuch that the ſum of every two of them may be a ſquare number.

SOLUTION.

Let x^2, y^2, and z^2, denote the numbers; then by the queſtion x^2+y^2, x^2+z^2, y^2+z^2 must all be rational ſquares.

Aſſume $x+y-a$, for the root of the firſt, and $x+z-b$, for the root of the ſecond, that is, put

$$x^2+y^2=(x+y-a)^2=x^2+y^2+2xy-2a(x+y)+a^2, \text{ and}$$
$$x^2+z^2=(x+z-b)^2=x^2+z^2+2xz-2b(x+z)+b^2;$$

from the former of which $x=\dfrac{a^2-2ay}{2(a-y)}$, and from the latter $x=\dfrac{b^2-2bz}{2(b-z)}$; therefore $\dfrac{a^2-2ay}{2(a-y)}=\dfrac{b^2-2bz}{2(b-z)}$; whence

$$z=\frac{by(2a-b)-ba(a-b)}{2y(a-b)+a(2b-a)};$$

put $a-b=d$, then $z=\dfrac{by(2a-b)-ba(a-b)}{2y(a-b)+a(2b-a)}=\dfrac{(a^2-d^2)y-ad(a-d)}{2dy+a^2-2da}$

and hence $y^2+z^2=y^2+\dfrac{\left\{(a^2-d^2)y-ad(a-d)\right\}^2}{(2dy+a^2-2da)^2}$

$$=\frac{y^2(2vd+a^2-2ad)^2}{(2dy+a^2-2ad)^2}+\frac{\left\{(a^2-d^2)y-ad(a-d)\right\}^2}{(2dy+a^2-2ad)^2}$$

which muſt be a ſquare by the queſtion, therefore

$$\left.\begin{matrix} y^2(2dy+a^2-2ad)^2 \\ +\left\{(a^2-d^2)y-ad(a-d)\right\}^2 \end{matrix}\right\}=\left\{\begin{matrix} 4y^4d^2+4y^3d(a^2-2ad)+y^2\left\{(a^2-2ad)^2+(a^2-d^2)^2\right\} \\ -2ady(a-d)(a^2-d^2)+a^2d^2(a-d)^2= \text{a square.} \end{matrix}\right.$$

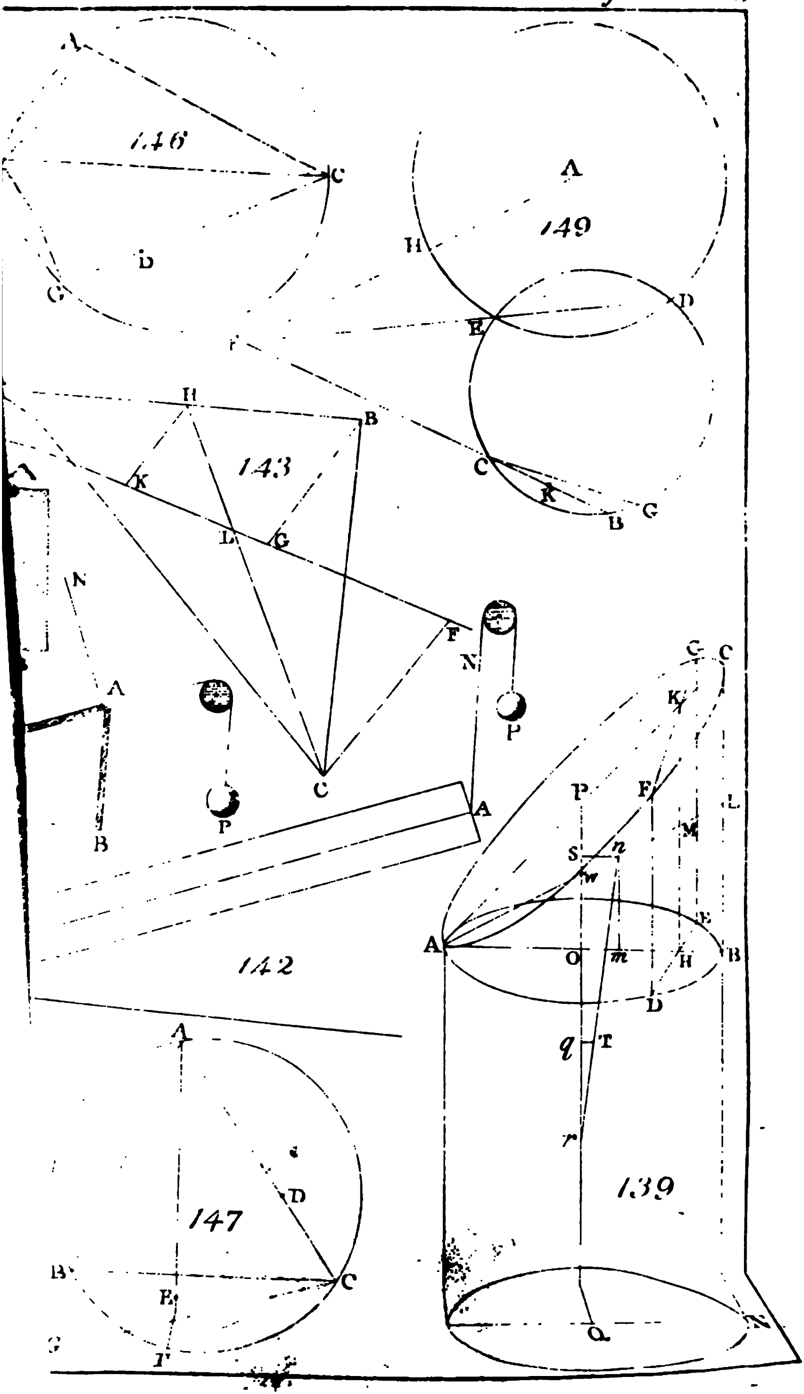
Aſſume $2y^2d+y(a^2-2ad)-ad(a-d)$ for its root; then

$$4y^4d^2+4y^3d(a^2-2ad)+y^2\left\{(a^2-2ad)^2+(a^2-d^2)^2\right\}$$
$$-2yad(a-d)(a^2-d^2)+a^2d^2(a-d)^2$$
$$=\left\{2y^2d+y(a^2-2ad)-ad(a-d)\right\}^2$$
$$=4y^4d^2+4y^3d(a^2-2ad)+y^2\left\{(a^2-2ad)^2-4ad^2(a-d)\right.$$
$$-2yad(a^2-2ad)(a-d)+a^2d^2(a-d)^2$$

which being reduced gives $y=\dfrac{2ad(2ad-d^2)}{(a+d)^2(a-d)+4ad^2}$, and by

reſtoring

Plate VIII. Fig. 138 to 151.



reſtoring the value of d, $y = \frac{2a\,(a-b)\,(a^2-b^2}{4a^3+b^3-4a^2b}$, where a and b may be taken at pleaſure (A). This I call the firſt formula for the value of y, and have marked it A.

Example 1. Suppoſe $a = 3$, and $b = 2$,

then, $y = \frac{15}{22}$; $x = \frac{a^2-2ay}{2\,(a-y)} = \frac{9-\frac{90}{22}}{2\,(3-\frac{15}{22})} = \frac{198-90}{2\,(66-15)} = \frac{108}{102} = \frac{18}{17}$;

and $z = \frac{b^2-2bx}{2\,(b-x)} = \frac{4-\frac{72}{17}}{2\,(2-\frac{18}{17})} = \frac{68-72}{2\,(34-18)} = \frac{-4}{32} = \frac{-1}{8}$.

Therefore $\frac{15}{22}$, $\frac{18}{17}$ and $\frac{1}{8}$ are the roots of three ſquares, the ſum of every two of which will be a rational ſquare. If whole numbers are deſired they may be obtained by reducing the fractions to a common denominator, and rejecting it; when the preceding fractions are ſo ordered there will be had $15 \times 17 \times 8 = 2040$; $18 \times 22 \times 8 = 3168$; and $22 \times 17 = 374$ for the roots of three ſquares that will anſwer; or becauſe theſe numbers are all diviſible by 2; 1020, 1584, and 187 are the roots of three ſquares that will anſwer.

Example 2. Take $a = 1$, and $b = -2$; then $y = \frac{-9}{2}$; $x = \frac{a^2-2ay}{2\,(a-y)} = \frac{10}{11}$; and $z = \frac{b^2-2bx}{2\,(b-x)} = \frac{-21}{16}$; from whence, by reducing the fractions to a common denominator, we ſhall obtain 792, 231, and 160 for the roots of three whole numbers, the ſum of the ſquares of every two of which will be a ſquare.

A ſecond formula for the value of y may be obtained as follows.

Let $2y^2d + y\,(a^2-2ad) + ad\,(a-d)$ be aſſumed for the root of the general expreſſion to be made a ſquare; that is, put

$$4y^4d^2 + 4y^3d\,(a^2-2ad) + y^2\left\{(a^2-2ad)^2 + (a^2-d^2)^2\right\}$$
$$-\,2yad\,(a-d)\,(a^2-d^2) + a^2d^2(a-d)^2$$
$$= \left\{2y^2d + y\,(a^2-2ad) + ad\,(a-d)\right\}^2$$
$$= 4y^4d^2 + 4y^3d\,(a^2-2ad) + y^2\left\{(a^2-2ad)^2 + 4ad^2(a-d)\right\}$$
$$+\,2yad\,(a-d)\,(a^2-2ad) + a^2d^2\,(a-d)^2$$

which being properly reduced gives $y = \frac{2ad\,(2a^2-2ad-d^2)}{(a+d)^2(a-d)-4ad^2}$;

and by reſtoring the value of d, $y = \frac{2a(a-b)\{4ab-(a^2+b^2)\}}{12a^2b+b^3-4a^3-8ab^2}$, (B) which is a ſecond formula, for the value of y, where a and b may, as before, be taken at pleaſure.

Example 1. Take $a = 2$ and $b = 1$, then $y = 12$; $x = \frac{a^2-2ay}{2(a-y)} = \frac{11}{5}$; and $z = \frac{b^2-2bx}{2(b-x)} = \frac{17}{12}$, which are the roots of three ſquares, the ſum of every two of which will be a ſquare; and theſe being ordered, as in the examples to formula A, will give $12 \times 5 \times 12 = 720$, $11 \times 12 = 132$, and $17 \times 5 = 85$ for the roots of three whole numbers that will anſwer.

Example 2. Take $a = 1$ and $b = -1$; then $y = \frac{24}{25}$; $x = \frac{a^2-2ay}{2(a-y)} = \frac{-23}{2}$; and $z = \frac{b^2-2bx}{2(b-x)} = \frac{-22}{21}$; and by ordering as before we ſhall have $24 \times 2 \times 21 = 1008$, $23 \times 25 \times 21 = 12075$, and $22 \times 25 \times 2 = 1100$ for the roots of three other whole numbers that will anſwer.

Again, aſſume $2y^2d - y(a^2-d^2) + ad(a-d)$ for the root of the expreſſion which is to be made a ſquare, that is, make

$$4y^4d^2 + 4y^3d(a^2-2ad) + y^2\{(a^2-2ad)^2 + (a^2-d^2)^2\}$$
$$- 2yad(a-d)(a^2-d^2) + a^2d^2(a-d)^2$$
$$= \{2y^2d - y(a^2-d^2) + ad(a-d)\}^2$$
$$= 4y^4d^2 - 4y^3d(a^2-d^2) + y^2\{(a^2-d^2)^2 + 4ad^2(a-d)\}$$
$$- 2yad(a-d)(a^2-d^2) + a^2d^2(a-d)^2,$$

from whence, by proper reduction, $y = \frac{4ad^2(a-d)-(a^2-2ad)^2}{4d(2a^2-2ad-d^2)}$, and reſtoring d, $y = \frac{a(8a^2b+4b^3-12ab^2-a^3)}{4(a-b)\{4ab-(a^2+b^2)\}}$ (C) which is a third formula for y.

Example 1. Take $b = 2$ and $a = 1$, then $y = \frac{1}{12}$, $x = \frac{a^2-2ay}{2(a-y)} = \frac{5}{11}$, and $z = \frac{b^2-2bx}{2(b-x)} = \frac{12}{17}$; and theſe being reduced to a common denominator as in the foregoing examples, and that denominator rejected, we ſhall have $1 \times 11 \times 17 = 187$,

$= 187$, $5 \times 12 \times 17 = 1020$, and $12 \times 12 \times 11 = 1584$ for the roots of three whole numbers that will anſwer, and are the ſame as were found in one of the preceding examples.

Example 2. Take $a = 4$, and $b = 1$; then $y = \frac{-20}{3}$, $x = \frac{a^2 - 2ay}{2(a - y)} = \frac{13}{4}$; and $z = \frac{b^2 - 2bx}{2(b - x)} = \frac{11}{9}$; theſe being reduced to the leaſt common denominator and that denominator rejected, there will be had 240, 117, and 44 for the roots of three whole numbers that will anſwer.

Laſtly, aſſume $2y^2 d + y(a^2 - d^2) - ad(a - d)$ for the root of the general expreſſion to be made a ſquare, or put

$$4y^4 d^2 + 4y^3 d(a^2 - 2ad) + y^2 \left\{ (a^2 - 2ad)^2 + (a^2 - d^2)^2 \right\} - 2yad(a - d)(a^2 - d^2) + a^2 d^2 (a - d)^2$$

$$= \left\{ (2y^2 d + y(a^2 - d^2) - ad(a - d) \right\}^2$$

$$= 4y^4 d^2 + 4y^3 d(a^2 - d^2) + y^2 \left\{ (a^2 - d^2)^2 - 4ad^2(a - d) \right\} - 2yad(a - d)(a^2 - d^2) + a^2 d^2 (a - d)^2$$

and this after proper reduction gives $y = \frac{(a^2 - 2ad)^2 + 4ad^2(a - d)}{4d^2(2a - d)}$, and reſtoring the value of d, $y = \frac{a(a^3 + 4b^3 - 4ab^2)}{4(a - b)(a^2 - b^2)}$ (D) and this is a fourth formula for y.

Example 1. Take $a = 2$ and $b = -1$; then $y = \frac{-2}{9}$, $x = \frac{a^2 - 2ay}{2(a - y)} = \frac{11}{10}$ and $z = \frac{b^2 - 2bx}{2(b - x)} = \frac{-16}{21}$; from whence, by proceeding as in the former examples, we ſhall obtain 140, 693, and 480 for the roots of three ſquares that will anſwer.

Example 2. Take $a = 1$, and $b = 2$; then $y = \frac{17}{12}$, $x = \frac{a^2 - 2ay}{2(a - y)} = \frac{11}{5}$, and $z = \frac{b^2 - 2bx}{2(b - x)} = 12$, and from hence we ſhall obtain 85, 132, and 720 for the roots of three ſquares that will anſwer.

Example 3. Take $a = 3$, and $b = 4$; then $y = \frac{39}{4}$, $x = \frac{a^2 - 2ay}{2(a - y)} = \frac{11}{3}$, and $z = \frac{b^2 - 2bx}{2(b - x)} = -20$, from whence we ſhall obtain 44, 117, and 240 for the roots of three numbers that will anſwer.

Example 4. Take $a = 3$, and $b = 1$, then $y = \frac{57}{64}$, $x = \frac{13}{15}$, and $z = \frac{11}{4}$, and from theſe we get 832, 855, and 2640 for the roots of three numbers that will anſwer.

There is already a pretty general anſwer to the preceding queſtion in No. 12 of the Old Series of the Repoſitory; but the ſolution here given is far more comprehenſive, as will appear from the great variety of examples which are annexed to illuſtrate the different formulæ. I believe it would be difficult to obtain ſome of the ſets of numbers which are deduced in the annexed examples by any other method of ſolution.

PROBLEM II.

To find values for the ſides of a triangle in whole numbers ſuch that the lengths of the three lines from the angles to the middle of the oppoſite ſides may be expreſſed by rational whole numbers.

SOLUTION.

Let ACB (fig. 150, pl. 7.) be a triangle, AE, BF, and CD lines from the angles to the middle of the oppoſite ſides. Put $AF = FC = x$, $BE = EC = y$, and $AD = DB = z$. Then by prop. 28, B. 2d. Emerſon's Geometry,

$(AC)^2 + (BC)^2 = 2\,(AD)^2 + 2\,(CD)^2$; whence

$\frac{(AC)^2 + (BC)^2}{2} - (AD)^2 = 2x^2 + 2y^2 - z^2 = (CD)^2 =$ a ſquare,

and in like manner $2x^2 + 2z^2 - y^2 = (AE)^2 =$ a ſquare;

and $2y^2 + 2z^2 - x^2 = (BF)^2 =$ a ſquare.

Put $x - y + n = z$, then

$$2x^2 + 2y^2 - z^2 = 2x^2 + 2y^2 - (x - y + n)^2$$
$$= x^2 + 2xy + y^2 - 2n\,(x - y) - n^2$$

= a ſquare. Aſſume $x + y - s$ for its root, that is, put

$$x^2 + 2xy + y^2 - 2n\,(x - y) - n^2 = (x + y - s)^2$$
$$= x^2 + 2xy + y^2 - 2s\,(x + y) + s^2;$$

from whence $x = \dfrac{2y(n+s)-(n^2+s^2)}{2(n-s)}$; and

$$z = x - y + n = \frac{2y(n+s)-(n^2+s^2)}{2(n-s)} - y + n = \frac{4sv+n^2-2sn-s^2}{2(n-s)}.$$

From what has been deduced it is manifeſt that we may take $x = 2v(n+s)-(n^2+s^2)$, $y = 2v(n-s)$ & $z = 4sv+n^2-2sn-s^2$, and by means of theſe

$$2x^2+2y^2-z^2 = \left\{ \begin{array}{l} 2\{2v(n+s)-n^2+s^2\}^2+8v^2(n-s)^2 \\ -(4sv+n^2-2sn-s^2)^2 \end{array} \right.$$

$$= (n^2+2ns-s^2)^2-8nv(n^2+2ns-s^2)+16n^2v^2$$

$$= (n^2+2ns-s^2-4nv)^2 = \text{a ſquare.}$$

Alſo

$$2x^2+2z^2-y^2 = \left\{ \begin{array}{l} 4s^4+8s^3n+8s^2n^2-8sn^3+4n^4 \\ -8v(3s^3+3s^2n-sn^2+n^3)+4v^2(3s+n)^2 \end{array} \right\} = \text{a ſquare;}$$

and dividing by 4

$$\left. \begin{array}{l} s^4+2s^3n+2s^2n^2-2sn^3+n^4 \\ -2v(3s^3+3s^2n-sn^2+n^3)+v^2(3s+n)^2 \end{array} \right\} = \text{a ſquare; \&}$$

$$2y^2+2z^2-x^2 = \left\{ \begin{array}{l} s^4+8s^3n+8s^2n^2-8sn^3+n^4 \\ -4v(3s^3+7s^2n-5sn^2-n^3)+4v^2(3s-n)^2 \end{array} \right\} = \text{a ſquare.}$$

And from theſe expreſſions we might obtain a general ſolution, but the calculation would be very troubleſome. In order therefore to avoid a tedious calculation we ſhall proceed in a manner leſs general.

By a bare inſpection of the latter of the two formulæ which remain to be made ſquares it will at once appear, that if we take $n = 3s$, then v will only have one dimenſion in that formula, and from thence we ſhall have

$$x = 2v(n+s)-(n^2+s^2) = 8sv-10s^2;$$

$$y = 2v(n-s) = 4sv \text{ and}$$

$$z = 4sv+n^2-2sn-s^2 = 4sv+2s^2.$$

But as each of theſe expreſſions is diviſible by $2s$, therefore we may put

$$x = 4v-5s,\ y = 2v, \text{ and } z = 2v+s;$$

and by means of theſe

$$2x^2+2y^2-z^2 = (6v-7s)^2 = \text{a ſquare,}$$

$$2x^2+2z^2-y^2 \left\{ \begin{array}{l} = 36v^2-72sv+52s^2 \\ = 4(9v^2-18sv+13s^2) = \text{a ſquare,} \end{array} \right.$$

and $2y^2+2z^2-x^2 = 48sv-23s^2$ = a ſquare.

Therefore

$9v^2-18sv+13s^2$ and $48sv-23s^2$ are to be ſquares.

Put the laſt expreſſion, *viz.* $48sv - 23s^2 = t^2$, whence $v = \frac{t^2 + 23s^2}{48s}$ which being written for v in the former expreſſion, we have

$$\frac{(t^2 + 23s^2)^2}{256s^2} - 6(t^2 + 23s^2) + 13s^2 = \frac{t^4 - 50t^2s^2 + 1649s^4}{256s^2} = \text{a ſquare};$$

therefore $t^4 - 50t^2 s^2 + 1649s^4 =$ a ſquare; and this will be the caſe when $t = s$. But this relation of t and s would make the ſum of two of the ſides equal to the remaining one, and therefore no triangle would be formed. In order to obtain ſuch a relation of t and s as will ſatisfy the queſtion, put $t = r + s$; then $t^4 - 50t^2s^2 + 1649s^4 = r^4 + 4r^3s - 44r^2s^2 - 96rs^3 + 1600s^4 =$ a ſquare. Aſſume $r^2 - \frac{6rs}{5} + 40s^2$ for its root, that is, put $r^4 + 4r^3s - 44r^2s^2 - 96rs^3 + 1600s^4 = (r^2 - \frac{6rs}{5} + 40s^2)^2 = r^4 - \frac{12r^3s}{5} + \frac{36r^2s^2}{25} + 80r^2s^2 - 96rs^3 + 1600s^4$, which after proper reduction gives $\frac{r}{s} = \frac{98}{5}$, wherefore r and s may be expounded by any numbers in the ratio of 98 to 5. Take $r = 98$ and $s = 5$, then $t = r + s = 103$; whence $v = \frac{t^2 + 23s^2}{48s} = \frac{233}{5}$, $x = 4v - 5s = \frac{932}{5} - 25 = \frac{807}{5}$, $y = 2v = \frac{466}{5}$ and $z = 2v + s = \frac{401}{5}$, and rejecting the common denominator 5, we may take $x = 807$, $y = 466$, and $z = 491$; and therefore the ſides of a triangle that will anſwer are 1614, 932, and 982.

The values of x, y, and z, deduced in the preceding ſolution will make the expreſſions $(x+y)^2 + (x-y)^2 - z^2$, $(x+z)^2 + (x-z)^2 - y^2$ & $(y+z)^2 + (y-z)^2 - x^2$ all rational ſquares; thoſe expreſſions being reſpectively equal to $2x^2 + 2y^2 - z^2$, $2x^2 + 2z^2 - y^2$ and $2y^2 + 2z^2 - x^2$.

PROBLEM III.

To find two iſoſceles triangles ſuch, that their perimeters and *areas* may be equal.

SOLUTION.

SOLUTION.

Let ACB, and DEF (fig. 151, pl. 7.) be two iſoſceles triangles; draw CD and FG perpendicular to AB and DE reſpectively.

Put $AC = BC = m^2 + n^2$, $AD = DB = m^2 - n^2$; then $2mn = CD$. Alſo put $DF = EF = p^2 + q^2$, $DG = GE = p^2 - q^2$; then $2pq = FG$; and becauſe the perimeters of the triangles are equal,

$2AC + 2AD = 2DF + 2DG$, that is, $4m^2 = 4p^2$, or $m = p$.

Again ſince the areas are equal

$DC \times AD = FG \times DG$, that is, $2mn\,(m^2 - n^2) = 2pq\,(p^2 - q^2)$; exterminating p by means of the equation $m \doteq p$, we have,

$2mn(m^2 - n^2) = 2mq\,(m^2 - q^2)$; whence $m^2 = \dfrac{n^3 - q^3}{n - q} = n^2 + nq + q^2$;

put $m = p = \dfrac{rn}{s} - q$, that is, $m^2 = (\dfrac{rn}{s} - q)^2 = \dfrac{r^2 n^2}{s^2} - \dfrac{2rnq}{s} + q^2 = n^2 + nq + q^2$;

from whence, after proper reduction, $n = \dfrac{q\,(2rs + s^2)}{r^2 - s^2}$.

Take $q = r^2 - s^2$, then $n = 2rs + s^2$ and $m = p = \dfrac{rn}{s} - q = r^2 + rs + s^2$;

whence $AC = m^2 + n^2 = r^4 + 2r^3 s + 7r^2 s^2 + 2rs^3 + 2s^4$

$AB = 2AD = 2\,(m^2 - n^2) = 2r\,(r + 2s)\,(r^2 - s^2)$,

$DF = p^2 + q^2 = 2r^4 + 2r^3 s + r^2 s^2 + 2rs^3 + 2s^4$,

$DE = 2\,DG = 2\,(p^2 - q^2) = 2rs\,(2r^2 + 5rs + 2s^2)$;

which are general expreſſions for the ſides of the two iſoſceles triangles.

Example 1. Take $r = 2$, and $s = 1$, then $AC = 74$, $AB = 48$; $DF = 58$, and $DE = 80$; and as theſe numbers are all diviſible by 2, we may take $AC = 37$, $AB = 24$; $DF = 29$, and $DE = 40$.

Example 2. Take $r = 3$, and $s = 1$, then $AC = 218$, $AB = 240$; $DF = 233$, and $DE = 210$.

ARTICLE XI.

Problems relating to the Twilight of Shorteſt Duration.

By Astronomicus.

PROBLEM I.

TO find the day when the twilight is ſhorteſt in a given latitude.

Let P (fig. 160, pl. 9.) be the elevated pole, Z the zenith, and NPZS the meridian: let pq be part of the ſmall circle where twilight ends (which is 18° below the horizon), and let H be the point of the horizon where the ſun riſes in the morning, and K the point where he paſſes the circle pq, bounding twilight, in the evening: then it is obvious that the difference of the horary angles SPH and SPK will be proportional to the duration of twilight.

Through the points H and K draw the great circle HOK; biſect the arc HK in m, and draw the great circle Pm; draw alſo the azimuth circles ZH, ZrK.

Suppoſing the ſun's declination to remain unchanged from the time of riſing in the morning to the time of paſſing the ſmall circle pq in the evening, the polar diſtances PH and PK will be equal to one another. In the triangle ZPH, we have (No. 2, Repoſitory, Part 3, page 11.)
$\text{Coſ. }ZH = \text{coſ. }PZ\,\text{coſ. }PH + \text{ſin. }PZ\,\text{ſin. }PH\,\text{coſ. }ZPH$; and in like manner, in the triangle KPZ, we have
$\text{coſ. }ZK = \text{coſ. }PK\,\text{coſ. }ZP + \text{ſin. }PK\,\text{ſin. }PZ\,\text{coſ. }KPZ.$

Take the difference of the equations juſt found, and
$\text{ſin. }PZ\,\text{ſin. }PK\,(\text{coſ. }HPZ - \text{coſ. }KPZ) = \text{coſ. }ZH - \text{coſ. }ZK$,
now $\text{coſ. }ZH = 0$, becauſe $ZH = 90^\circ$, and $-\text{coſ. }ZK = -\text{coſ. }108^\circ = \text{ſin. }18^\circ$: alſo $\text{coſ. }HPZ - \text{coſ. }KPZ =$
$$2 \times \text{ſin.}\,\frac{KPZ + HPZ}{2} \times \text{ſin.}\,\frac{KPZ - HPZ}{2}$$
; therefore we have
$$\text{ſin. }PZ\,\text{ſin. }PK\,\text{ſin.}\,\frac{KPZ + HPZ}{2}\,\text{ſin.}\,\frac{KPZ - HPZ}{2} = \tfrac{1}{2}\,\text{ſin. }18^\circ.$$

Becauſe $KP = PH$ and $Km = mH$, it is obvious that the great circle Pm will cut the great circle HK at right angles, and further that the angle KPm = angle mPH. Therefore angle

$KPm = \frac{KPZ + KPH}{2}$. And, on account of the right angled triangle KPm, $\text{ſin. } KP \text{ ſin. } KPm = \text{ ſin. } Km = \text{ ſin. } \frac{KH}{2} = \text{ ſin. } KP \text{ ſin. } \frac{KPZ + HPZ}{2}$. Therefore we ſhall have

$$\text{ſin. } PZ \text{ ſin. } \frac{HK}{2} \text{ ſin. } \frac{KPZ - HPZ}{2} = \tfrac{1}{2} \text{ ſin. } 18^\circ.$$

Now $HK = HO + OK$; and it is manifeſt that HO is a ſemicircle; therefore $HK = 180^\circ + OK$; and $\text{ſin. } \frac{HK}{2} = \text{ ſin. } (90^\circ + \frac{OK}{2}) = \text{ coſ. } \frac{OK}{2}$: therefore we have

$$\text{ſin. } PZ \text{ coſ. } \frac{OK}{2} \text{ ſin. } \frac{KPZ - HPZ}{2} = \tfrac{1}{2} \text{ ſin. } 18^\circ.$$

The difference of the horary angles KPZ and HPZ (and, conſequently, the duration of twilight) will be a minimum when $\text{ſin. } \frac{KPZ - HPZ}{2}$ is a minimum: but from the laſt equation $\text{ſin. } \frac{KPZ - HPZ}{2}$ will be a minimum when $\text{coſ. } \frac{KO}{2}$ is a maximum; for all the other quantities in that equation are conſtant. Therefore, it is plain, that the duration of twilight is a minimum, when the arch KO is a minimum, that is, when the great circle HOK is perpendicular to the horizon, or when it paſſes through Z, the pole of the horizon.

Thus the duration of twilight is a minimum in any latitude, on that day when the ſun croſſes the ſmall circle bounding twilight, after ſun-ſetting, on the ſame azimuth circle on which he roſe in the horizon in the morning; or, which is the ſame thing, when the ſun croſſes the ſmall circle bounding twilight, before ſun-riſing, on the ſame azimuth circle on which he will ſet in the horizon in the evening.

It is now eaſy to find the ſun's declination (and, conſequently, the day of the year) when the twilight is a minimum: for ſuppoſe KZH to be a great circle paſſing through the zenith and the points H and K, as has been ſhewn to be the caſe when the twilight is the leaſt of all; and draw $P\mu$ perpendicular to that great circle: then becauſe $PK = PH$, therefore $K\mu = \mu H$. Therefore $\mu K = \frac{KZH}{2} = 99^\circ$ and $Z\mu = 9^\circ$: and in the triangle KZP, we have

$$\text{coſ. } Z\mu : \text{coſ. } \mu K :: \text{coſ. } PZ : \text{coſ. } PK:$$

Therefore $$\text{coſ. } PK = \frac{\text{coſ. } \mu K \ \text{coſ. } PZ}{\text{coſ. } \mu Z},$$

or, ſin. ☉'s declin. = — tan. 9° × ſin. latitude, where the ſign (—) ſhews that the declination is ſouth when the latitude is north, and *vice verſa*.

To find the length of the ſhorteſt twilight: in the right-angled triangle $ZP\mu$, we have

$$\text{ſin. } ZP\mu = \frac{\text{ſin. } Z\mu}{\text{ſin. } PZ} = \frac{\text{ſin. } 9^\circ}{\text{coſ. latitude}}$$

and the double of the angle $ZP\mu$, being converted into time, will give the duration of the ſhorteſt twilight required.

The ſame reſult may be obtained by putting 18° (which is the leaſt value of KO) for KO in the equation obtained above.

PROBLEM II.

To find the latitude of that place where the twilight is a minimum on a given day.

Let Z (fig. 161, pl. 9.) be zenith, NS the horizon, and H the point in the horizon where the ſun ſets; alſo let K be the point where the ſun, after ſetting, croſſes the ſmall circle $\gamma\tau$ bounding twilight: let P be the pole, and draw the circles of declination PH and PK and the azimuth circles ZH and ZK: laſtly, let a great circle paſs through H and K.

Then becauſe the day of the year is given the ſun's declination will be given: therefore the polar diſtance PH = PK, will be given: therefore in the triangle HPK, of which the two ſides HP and PK are conſtant, the leſs the baſe HK is, the leſs will be the horary angle HPK, which meaſures the duration of twilight; and when the baſe HK is leaſt of all, the angle HPK will likewiſe be leaſt of all. But in the triangle KZH, the ſide HK can never be leſs than the difference of the two ſides ZK and ZH, that is, HK can never be leſs than KL, or 18°. Therefore HK will be leaſt of all when it is equal to KL, or 18°. Therefore the duration of twilight will be leaſt of all at that place where the pole is ſo ſituated (as at *p*,) that the point H may coincide with the point L, or ſo that *p*K and *p*L may be each equal to the diſtance of the ſun from the elevated pole on the given day.

Thus the duration of twilight is a minimum on a given day at that place where the ſun croſſes the ſmall circle 18° below the horizon, after ſun-ſetting, on the ſame azimuth circle on which he ſet in the horizon: or, which is the ſame thing, where the ſun croſſes the ſmall circle 18° below the horizon, before ſun-riſing, *on the ſame azimuth* circle on which he will riſe in the horizon.

To find the latitude draw pm perpendicular to ZK: then becauſe $pK = pL$, therefore $Km = Lm$: therefore $Km = 9°$ and $mZ = 99°$. In the triangle pZK,

$$\text{coſ. } mK : \text{coſ. } mZ :: \text{coſ. } Kp : \text{coſ. } pZ,$$

$$\text{therefore coſ. } pZ = \frac{\text{coſ. } 99°}{\text{coſ. } 9°} \times \text{coſ. } pK = -\text{tan. } 9° \times \text{coſ. } pK$$

or, ſin. latitude = — tan. 9° × ſin. ☉'s declination, where the ſign (—) ſhews that the latitude is north when the ſun's declination is ſouth, and *vice verſa*.

To find the length of twilight on a given day at that place where it is the leaſt of all; we have

$$\text{ſin. } Kpm = \frac{\text{ſin. } Km}{\text{ſin. } pK} = \frac{\text{ſin. } 9°}{\text{coſ. ☉'s declination}},$$

and the double of the angle Kpm, converted into time, will give the duration of the ſhorteſt twilight on the given day.

The very leaſt twilight over all the earth, and for all the year, is at the equator, when the ſun is in the equinoctial points.

Take ſin. q = tan. 9° × ſin. 23° 28′; and deſcribe two ſmall circles on the earth, on oppoſite ſides of the equator, and at the diſtance q from it: then, at every place comprehended in the zone between the two circles, the twilight will, twice in a year, be leſs than at any other place on the ſurface of the earth.

The Editor has alſo been favoured with papers on this ſubject from two other correſpondents: *viz.* one from Mr. Benjamin Gompertz, containing the corrections of ſome inaccuracies in Emerſon's Solution to prob. 54 of his Miſcellanies; and the other from Mr. Surtees, containing the inveſtigation of both caſes of the general problem, and the application of the latter caſe to the ſolution of the 316th queſtion of the Old Series of the Repoſitory.

After what has been done above, it will be unneceſſary to give either of theſe papers at full length; we ſhall therefore only inſert that part which relates to the 316th queſtion.

Mr. Surtees, by putting m and t for the ſine and tangent of half the depreſſion (9°), c for the ſine of the latitude, and y for the ſine of the declination; finds $c = -ty$, when the declination is conſtant, and the duration of twilight the ſhorteſt poſſible, and the ſine of half the duration $= \frac{m}{\sqrt{(1 - y^2)}}$; which preciſely agrees with what is determined by Aſtronomicus in the problems above.

Now in the 316th queſtion, the ſun's declination on October 12th is 7° 9′ and $c = -ty =$ ſine of 1° 7′ $\frac{2}{3}$, the latitude of the place where the duration of twilight is abſolutely a minimum on that day; and the ſine of half the duration is $= \frac{m}{\surd(1 - y^2)}$ = ſine of 9° 4′ $\frac{2}{3}$, or its double = 18° 9′ $\frac{1}{3}$ = 1 *h* 12 $\frac{2}{5}$ *m*, the whole time of the duration of twilight in that latitude on the 12th of October.

To find the duration of twilight at the equator on any day of the year. Let P,S, (fig. 162, pl. 9.) repreſent the poles, PAHS the horizon, *d*H*d* a parallel of declination, and ZACN an azimuth circle paſſing through C, the place of the ſun when 18° below the horizon. Then ſin. PC = coſ. declination $= \surd(1 - y^2)$, and the ſine of the depreſſion, AC $= 2m\,\surd(1 - m^2)$; and in the right angled ſpherical triangle PAC, ſin. PC : rad. :: ſin. AC : ſin. ∠ APC $= \frac{2m}{\surd(1 - y^2)} \times \surd(1 - m^2)$, the ſine of the whole duration at the equator, the ſine of the ſun's declination being y.

Now the ſine of half the duration, found above, is $= \frac{m}{\surd(1 - y^2)}$; or the ſine of the whole duration $= \frac{2m}{\surd(1 - y^2)} \times \surd(1 - \frac{m^2}{1 - y^2})$ and therefore the ſine of the duration at the equator will be to the ſine of the duration in latitude $c = -ty$, as $\frac{2m}{\surd(1 - y^2)} \times \surd(1 - m^2)$ to $\frac{2m}{\surd(1 - y^2)} \times \surd(1 - \frac{m^2}{(1 - y^2)}$, or as $\surd(1 - m^2)$ to $\surd(1 - \frac{m^2}{1 - y^2})$; where it is evident, that if y be of any value whatever, the latter expreſſion will be leſs than the former, and therefore the duration of twilight in latitude $c = -ty$ is leſs than the time of duration at the equator. Conſequently the duration of twilight is *not* a minimum at the equator at *all* times of the year, but only on the days of the equinoxes when the ſine of half the duration $\frac{m}{\surd(1 - y^2)}$ is = to m, or the whole time = 1 hour and 12 minutes.

The error in finding the duration of twilight at the equator in the ſolution of the 316th queſtion ariſes from conſidering the triangle PHC as a right angled great circle ſpherical triangle, when it is really an iſoſceles triangle, and the arc HC a portion of a ſmall circle of the ſphere inſtead of a great one.

ARTICLE

ARTICLE XII.

Certain Fluents expreſſible by an Elliptic Arch.

By Mr. James Cunliffe, *R. M. College.*

THE fluent of $\dot{v}\sqrt{\left(\frac{1-e^2v^2}{1-v^2}\right)}$ is known to expreſs the length of an elliptic arch intercepted between the conjugate axis and a parallel ordinate at the diſtance v from the centre: the tranſverſe axis of the ellipſe being 1, and eccentricity e.

This expreſſion may, by various artifices, be transformed into others, the fluents whereof are, in like manner, expreſſible by the arch of an ellipsis.

Let us try the reſult of one artifice, that is, let us make the denominator $1-v^2$ a rational ſquare. For this purpoſe aſſume $1-vx$ for the root of $1-v^2$, that is, put $1-v^2 =$ $(1-vx)^2 = 1-2vx+v^2x^2$; whence $v = \frac{2x}{1+x^2}$,

$$\dot{v} = \frac{2\dot{x}(1-x^2)}{(1+x^2)^2},\ \sqrt{\left(\frac{1-e^2v^2}{1-v^2}\right)} = \sqrt{\frac{1-\frac{4e^2x^2}{(1+x^2)^2}}{1-\frac{4x^2}{(1+x^2)^2}}} = \frac{1}{1-x^2}\times\sqrt{\left\{1-x^2\times(4e^2-2)+x^4\right\}},$$

And, by means of theſe,

$$\dot{v}\sqrt{\left(\frac{1-e^2v^2}{1-v^2}\right)} = \frac{2\dot{x}}{(1+x^2)^2}\times\sqrt{\left\{1-x^2\times(4e^2-2)+x^4\right\}}.$$

Therefore the fluent of any expreſſion that can be reduced to the form $\frac{2\dot{x}}{(1+x^2)^2}\times\sqrt{\left\{1-x^2\,(4e^2-2)+x^4\right\}}$ may be exhibited by means of an elliptic arch.

As an example of the uſe of the foregoing concluſion, let us take the 77th queſtion of the Repoſitory, where it is required to find the fluent of $\dot{x}\sqrt{\frac{1+x^6}{(1+x^2)^5}}$.

In the firſt place $\dot{x}\sqrt{\frac{1+x^6}{(1+x^2)^5}} = \frac{\dot{x}}{(1+x^2)^2}\sqrt{\frac{1+x^6}{1+x^2}} = \frac{\dot{x}}{(1+x^2)^2} \times \sqrt{(1-x^2+x^4)}$.

Now by making $4e^2-2=1$, or, taking $e=\frac{1}{2}\sqrt{3}$, the two expreſſions become identical, that is, the fluent of $\dot{x}\sqrt{\frac{1+x^6}{(1+x^2)^5}}$ is expreſſible by half the length of the arch of an ellipſe intercepted between the conjugate axis and a parallel ordinate at the diſtance of $\frac{2x}{1+x}$, from the centre: the ſemi-tranſverſe axis of the ellipſe being 1, and eccentricity $\frac{1}{2}\sqrt{3}$, or the tranſverſe and conjugate axes of the ellipſe being 2 and 1 reſpectively.

And in general the fluent of $\frac{\dot{x}}{(1+x^2)^2}\sqrt{(1-ax^2+x^4)}$ is expreſſible by the arch of an ellipſe; where a may denote any poſitive number whatever; and even when a denotes a negative number leſs than 2.

Again, let it be required to find the fluent of $\frac{\dot{x}}{(1+x^2)^2}\sqrt{(1+x^4)}$.

It has been ſhewn that the fluent of $\frac{\dot{x}}{(1+x^2)^2}\times\sqrt{\left\{1-x^2\times(4e^2-2)+x^4\right\}}$ is expreſſible by half the length of the arch of an ellipſis intercepted between the conjugate axis and a parallel ordinate at the diſtance $\frac{2x}{1+x^2}$, from the centre: the ſemi-tranſverſe axis of the ellipſis being 1 and eccentricity e. And it is obvious that if we put $4e^2-2=0$, or $e=\sqrt{\frac{1}{2}}$, the ſecond term under the radical ſign will diſappear, and the expreſſion then becomes $\frac{\dot{x}}{(1+x^2)^2}\times\sqrt{(1+x^4)}$; and therefore its fluent is expreſſible by half the length of the arch of an ellipſis intercepted between the conjugate axis, and a parallel ordinate at the diſtance $\frac{2x}{1+x^2}$ from the centre; the ſemi-tranſverſe axis of the ellipſis being 1 and eccentricity $\sqrt{\frac{1}{2}}$, or the tranſverſe and conjugate axes of the ellipſe are 2, and $\sqrt{2}$ reſpectively.

To find the fluxion of the length of the arch of a given ellipſe, at any point of the curve in terms of the ſemi-diameter at that point.

Let a denote the ſemi-tranſverſe, and b the ſemi-conjugate axes of the ellipſe: y an ordinate thereof parallel to the conjugate axis at the diſtance x from the centre, and z the arch of the ellipſe intercepted between the conjugate axis and ordinate y.

Then it is well known, that

$$\dot{z}=\dot{x}\sqrt{\left\{\frac{a^2-\left(\frac{a^2-b^2}{a^2}\right)x^2}{a^2-x^2}\right\}}.$$

Now let v denote the ſemi-diameter at the point where the ordinate y cuts the curve; then we ſhall have

$$v^2=x^2+y^2=x^2+\frac{b^2}{a^2}\times(a^2-x^2),$$

from whence

$$x^2\times\left(\frac{a^2-b^2}{a^2}\right)=v^2-b^2;\ a^2-\left(\frac{a^2-b^2}{a^2}\right)\times x^2=a^2+b^2-v^2;$$

$$a^2-x^2=a^2-\frac{a^2\times(v^2-b^2)}{a^2-b^2}=\frac{a^2}{a^2-b^2}\times(a^2-v^2);$$

$$\text{also } \dot{x}=\frac{a}{\sqrt{(a^2-b^2)}}\times\frac{v\dot{v}}{\sqrt{(v^2-b^2)}}.$$

And from hence,

$$\dot{z}=\dot{x}\sqrt{\left\{\frac{a^2-\left(\frac{a^2-b^2}{a^2}\right)\times x^2}{a^2-x^2}\right\}}=v\dot{v}\sqrt{\left\{\frac{a^2+b^2-v^2}{(v^2-b^2)\times(a^2-v^2)}\right\}}$$

Q. E. I.

Put $cd - cu = v^2$, then

$$\dot{v}\sqrt{\left\{\frac{a^2+b^2-v^2}{(v^2-b^2)(a^2-v^2)}\right\}} = \frac{-\dot{u}\sqrt{c}}{2}\sqrt{\left\{\frac{\frac{a^2+b^2}{c}-d+u}{(d-\frac{b^2}{c}-u)(\frac{a^2}{c}-d+u)}\right\}}.$$

Whence it appears, that fluxionary expreſſions of the form $\dot{u}\sqrt{\left\{\frac{m+u}{(r-u)(s+u)}\right\}}$ may be integrated by means of an elliptic arch; where m, r, and s, denote conſtant quantities.

Again, put $d-\frac{b^2}{c}=\frac{a^2}{c}-d$, which gives $2d=\frac{a^2+b^2}{c}$, and

$$\frac{-\dot{u}\sqrt{c}}{2}\sqrt{\left\{\frac{\frac{a^2+b^2}{c}-d+u}{(d-\frac{b^2}{c}-u)(\frac{a^2}{c}-d+u)}\right\}} = \frac{-\dot{u}\sqrt{c}}{2}\sqrt{\left\{\frac{d+u}{(d-\frac{b^2}{c})^2-u^2}\right\}}$$

the fluent of which is expreſſible by the arch of an ellipſe intercepted between the conjugate axis and a parallel ordinate at the diſtance

$$x=\frac{a\sqrt{(v^2-b^2)}}{\sqrt{(a^2-b^2)}}=\frac{a}{\sqrt{(a^2-b^2)}}\times\sqrt{(cd-b^2-cu)}$$

from the centre, the ſemi-tranſverſe, and ſemi-conjugate axes of the ellipſe being a, $=\sqrt{(2dc-b^2)}$, and b reſpectively.

ARTICLE

NEW SERIES
OF THE
MATHEMATICAL REPOSITORY.

VOL. I. PART III.

MATHEMATICAL MEMOIRS, EXTRACTED FROM WORKS OF EMINENCE.

ARTICLE I.

Solutions of ſome Problems relative to Spherical Triangles; together with a complete Analyſis of theſe Triangles.

By I. L. Lagrange.*

IT is known that in rectilineal triangles the ſides are proportional to the ſines of the oppoſite angles, and it is eaſily demonſtrated that this conſtant ratio of the ſides to the ſines is equal to the diameter of the circumſcribing circle.

The ſame ratio may alſo be expreſſed by the area of the triangle, and it may be eaſily proved to be equal to the product of the three ſides divided by twice the area.

1. But in order to expreſs that ratio by the ſides of the triangle alone, it is only neceſſary to conſider that, by putting a, b, c for the three ſides, and A, B, C for the angles oppoſite to them, we have by a known theorem $a^2 = b^2 + c^2 - 2bc$ coſ. A;

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* From Journal De l'ecole Polytechnique, Vol. II. Page 270.

therefore coſ. A $= \dfrac{b^2 + c^2 - a^2}{2bc}$, and hence

$$\text{ſin.}\,A = \frac{\sqrt{[4b^2c^2 - (b^2 + c^2 - a^2)^2]}}{2bc}.$$

Therefore, if in order to ſimplify, we put

$$d = \sqrt{[4b^2c^2 - (b^2 + c^2 - a^2)^2]}$$

we have ſin. A $= \dfrac{d}{2bc}$, and hence $\dfrac{a}{\text{ſin.}\,A} = \dfrac{2abc}{d}$ which fraction expreſſes the ratio ſought.

If r be put for the radius of the circle which circumſcribes the triangle, and s for the ſurface, or area of the ſame triangle, we have $\dfrac{a}{\text{ſin.}\,A} = \dfrac{2abc}{d} = 2r = \dfrac{abc}{2s}$; therefore $r = \dfrac{abc}{d}$, and and $s = \dfrac{abc}{4r} = \dfrac{d}{4}$.

2. By developing the ſquare of $b^2 + c^2 - a^2$, and reducing, we have

$$d = \sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)}$$

a formula in which the three ſides a, b, c are equally concerned, as they ought to be.

But this formula may be rendered more ſimple and more commodious for logarithmic calculation by reſolving it into factors. For

$$4b^2c^2 - (b^2 + c^2 - a^2)^2 = (2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2),$$

and again reſolving each of theſe factors into two, we have

$$d = \sqrt{[(a + b + c)(a - b + c)(a + b - c)(-a + b + c)]}.$$

Theſe formulas are already known, and I only give them here as an introduction to inveſtigations which are to follow.

3. Since in ſpherical triangles the ſines of the ſides are proportional to the ſines of the angles oppoſite to them, we may be deſirous to inveſtigate the conſtant ratio, and to diſcover if it depends, as in rectilineal triangles, upon the radius of the circumſcribed circle, or the area of the triangle.

Let us denote as before the three ſides of a ſpherical triangle by a, b, c, and the three angles oppoſite to thoſe ſides by A, B, C.

Then by a known theorem

$$\text{coſ.}\,a = \text{coſ.}\,b\ \text{coſ.}\,c + \text{ſin.}\,b\ \text{ſin.}\,c\ \text{coſ.}\,A;$$

therefore coſ. A $= \dfrac{\text{coſ.}\,a - \text{coſ.}\,b\ \text{coſ.}\,c}{\text{ſin.}\,b\ \text{ſin.}\,c}$; and hence

$$\text{ſin.}\,A = \frac{\sqrt{[\text{ſin.}^2 b\ \text{ſin.}^2 c - (\text{coſ.}\,a - \text{coſ.}\,b\ \text{coſ.}\,c)^2]}}{\text{ſin.}\,b\ \text{ſin.}\,c}$$

Let us, to abreviate, put

$$f = \sqrt{[\text{ſin.}^2 b\, \text{ſin.}^2 c - (\text{coſ.}\, a - \text{coſ.}\, b\, \text{coſ.}\, c)^2]}$$

thus we have ſin. A $= \frac{f}{\text{ſin.}\, b\, \text{ſin.}\, c}$, and therefore

$$\frac{\text{ſin.}\, a}{\text{ſin.}\, A} = \frac{\text{ſin.}\, a\, \text{ſin.}\, b\, \text{ſin.}\, c}{f}$$

an expreſſion for the ratio ſought analogous to that which has been already found for rectilineal triangles (No. 1).

As the quantity f is expreſſed by a ſquare root, and may, therefore, have either the ſign plus, or minus prefixed to it, we remark, that relatively to ſpherical triangles, it ought to be always poſitive; for the ſides and the angles of every triangle being always leſs than two right angles, the ſines are neceſſarily always poſitive.

4. The radical quantity f is alſo ſuſceptible of reductions ſimilar to thoſe of No. 2. For by ſubſtituting inſtead of ſin.^{2} b and ſin.^{2} c their values $1 - \text{coſ.}^2 b$ and $1 - \text{coſ.}^2 c$; and afterwards reducing, we have

$$f = \sqrt{[1 - \text{coſ.}^2 a - \text{coſ.}^2 b - \text{coſ.}^2 c + 2\,\text{coſ.}\, a\, \text{coſ.}\, b\, \text{coſ.}\, c]}$$

where it appears that the three ſides a, b, c are alike concerned.

In like manner the quantity under the radical ſign may be reſolved into factors, for we have preſently

$$\begin{aligned} &\text{ſin.}^2 b\, \text{ſin.}^2 c - (\text{coſ.}\, a - \text{coſ.}\, b\, \text{coſ.}\, c)^2 \\ &= (\text{ſin.}\, b\, \text{ſin.}\, c + \text{coſ.}\, a - \text{coſ.}\, b\, \text{coſ.}\, c) \\ &\quad \times (\text{ſin.}\, b\, \text{ſin.}\, c - \text{coſ}\, a + \text{coſ.}\, b\, \text{coſ.}\, c) \\ &= [\text{coſ.}\, a - \text{coſ.}\, (b + c)]\, [\text{coſ.}\, (b - c) - \text{coſ.}\, a] \end{aligned}$$

but we have, in general $\text{coſ.}\, a - \text{coſ.}\, h = 2\, \text{ſin.}\, \frac{a+h}{2} \times \text{ſin.}\, \frac{h-a}{2}$

therefore, reſolving thus the two factors, we have

$$f = 2\sqrt{[\text{ſin.}\frac{a+b+c}{2} \times \text{ſin.}\frac{a-b+c}{2} \times \text{ſin.}\frac{a+b-c}{2} \times \text{ſin.}\frac{-a+b+c}{2}]}$$

an expreſſion very convenient for logarithmic calculation.

5. Let us now ſeek the radius of the circle circumſcribing the ſpherical triangle. It is evident that this circle muſt be a ſmall circle of the ſphere, and that it will alſo circumſcribe the rectilineal triangle formed by the chords of the arches a, b, c; but theſe chords being expreſſed by $2\, \text{ſin.}\, \frac{a}{2}$, $2\,\text{ſin.}\, \frac{b}{2}$, $2\,\text{ſin.}\, \frac{2}{c}$, it is only neceſſary to ſubſtitute theſe quantitities inſtead of a, b,

c in the expreſſion of the radius r (No. 2), that is to ſay, in $\frac{abc}{d}$.

Let us put R for the radius of the circumſcribed circle, and h for the value which d acquires by ſubſtituting for a, b, c as above; thus we have

$$R = \frac{8\,\text{ſin.}\,\frac{a}{2} \times \text{ſin.}\,\frac{b}{2} \times \text{ſin.}\,\frac{c}{2}}{h}.$$

But we have $h^2 = 32\,(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{b}{2})^2 + 32\,(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{c}{2})^2 +$ $32\,(\text{ſin.}\,\frac{b}{2}\,\text{ſin.}\,\frac{c}{2})^2 - 16\,(\text{ſin.}\,\frac{a}{2})^4 - 16(\text{sin.}\,\frac{b}{2})^4 - 16(\text{sin.}\,\frac{c}{2})^4$; and ſubſtituting for $2(\text{sin.}\,\frac{a}{2})^2$, $2(\text{sin.}\,\frac{b}{2})^2$, $2(\text{sin.}\,\frac{c}{2})^2$ their values $1 - \text{coſ.}\,a$, $1 - \text{coſ.}\,b$, $1 - \text{coſ.}\,c$ we have after proper reductions

$$\begin{aligned} h^2 = 12 &- 8\,\text{coſ.}\,a - 8\,\text{coſ.}\,b - 8\,\text{coſ.}\,c \\ &+ 8\,\text{coſ.}\,a\,\text{coſ.}\,b + 8\,\text{coſ.}\,a\,\text{coſ.}\,c + 8\,\text{coſ.}\,b\,\text{coſ.}\,c \\ &- 4\,\text{coſ.}^2\,a - 4\,\text{coſ.}^2\,b - 4\,\text{coſ.}^2\,c \end{aligned}$$

which expreſſion may be reduced to the following

$$4f^2 + 8(1 - \text{coſ.}\,a)\,(1 - \text{coſ.}\,b)\,(1 - \text{coſ.}\,c)$$

that is to ſay, to

$$4f^2 + 64\,(\text{ſin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2})^2$$

by ſubſtituting the value of f of No. 4. Thus we have

$$R = \frac{4\,\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2}}{\sqrt{[f^2 + 16(\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2})^2]}}.$$

6. Now by conſidering the radius of the ſphere which paſſes through the centre of the ſmall circle circumſcribing the triangle, it is evident that this radius is perpendicular to the plane of the circle, and that it meets the ſurface of the ſphere at a point which is the pole of the ſame circle.

Therefore, putting φ for the arch which meaſures the diſtance of the circle from its pole, we have evidently $R = \text{sin.}\,\varphi$, therefore

$$\text{sin. }\varphi = \frac{4\text{ sin. }\frac{a}{2}\text{ sin. }\frac{b}{2}\text{ sin. }\frac{c}{2}}{\sqrt{[f^2 + 16(\text{sin. }\frac{a}{2}\text{ sin. }\frac{b}{2}\text{ sin. }\frac{c}{2})^2]}}.$$

from whence we derive

$$\text{coſ. }\varphi = \frac{f}{\sqrt{[f^2 + 16(\text{ſin. }\frac{a}{2}\text{ ſin. }\frac{b}{2}\text{ ſin. }\frac{c}{2})^2]}},$$

and hence

$$\text{tan. }\varphi = \frac{4\text{ sin. }\frac{a}{2}\text{ sin. }\frac{b}{2}\text{ sin. }\frac{c}{2}}{f}.$$

Wherefore, ſeeing that $\text{sin. } a = 2\text{ sin. }\frac{a}{2}\text{ coſ. }\frac{a}{2}$, and ſo on with reſpect to sin. b and sin. c, we have (No. 3).

$$\frac{\text{sin. } a}{\text{sin. A}} = 2\text{ tan. }\varphi\text{ coſ. }\frac{a}{2}\text{ coſ. }\frac{b}{2}\text{ coſ. }\frac{c}{2}.$$

7. If it were required to find the area of the rectilineal triangle formed by the chords of the arches a, b, c and inſcribed in the circle we have juſt now been conſidering; by putting S for that area, it would only be neceſſary to change, in the formula $s = \frac{abc}{4r}$ of No. 1, s into S, r into R, and a, b, c into $2\text{ sin. }\frac{a}{2}$; $2\text{ sin. }\frac{b}{2}$, $2\text{ sin. }\frac{c}{2}$ thus we ſhould have immediately

$$\text{S} = \frac{2\text{ sin. }\frac{a}{2}\text{ sin. }\frac{b}{2}\text{ sin. }\frac{c}{2}}{\text{R}}$$

or, becauſe R = sin. φ (No. 6)

$$\text{S} = \frac{2\text{ sin. }\frac{a}{2}\text{ sin. }\frac{b}{2}\text{ sin. }\frac{c}{2}}{\text{sin. }\varphi}.$$

If now we conſider the triangular pyramid which has this triangle for its baſe, and of which the vertex is at the centre of the ſphere, it is evident that the height of the pyramid is coſ. φ, therefore its ſolidity is $\frac{\text{S coſ. }\varphi}{3}$, or, ſubſtituting for S the value we have already found,

2 sin.

$$\frac{2\sin.\frac{a}{2}\sin.\frac{b}{2}\sin.\frac{c}{2}}{3\tan.\,\varphi}$$

and again, by ſubſtituting for tan. φ its value already ſound (No. 6), we have $\frac{f}{6}$ for the ſolidity of the pyramid.

8. It remains for us to conſider the area of the ſpherical triangle itſelf formed by the arcs a, b, c. There is a beautiful and known theorem, according to which, the area of a ſpherical triangle is to the whole ſurface of the ſphere, as the exceſs of the three angles of the triangle above two right angles to eight right angles. This theorem is commonly attributed to Albert Girard, who gave it in ſubſtance, in a work entitled *Invention Nouvelle en Algebre*, printed at Amſterdam in 1629; but as the proof which he gives is not rigorous, and cannot even be conſidered as an induction, the theorem ought rather to be attributed to Cavallerius, who gave it in the *Directericum generale uranometricum*, printed at Bologna in 1632, with the elegant demonſtration firſt publiſhed by Wallis, and ſince inſerted in moſt works upon trigonometry.

Let Σ denote the exceſs of the three angles of the triangle above two right angles, then, retaining the notation hitherto employed, and putting D for a right angle,

$$\Sigma = A + B + C - 2D.$$

Thus the area of the triangle of which the sides are a, b, c and the angles A, B, C ſhall be the $\frac{\Sigma}{8D}$ part of the whole ſurface of the ſphere; and if this ſurface be conſidered as equal to 8D, we may take Σ for the area of the triangle.

9. If we ſuppoſe the sides b and c, which contain the angle A, to be produced till each of them be a quadrant, the angles B, C will then become right angles, and the side a will become equal to the oppoſite angle A; then the area of the iſoſceles right angled triangle becomes A; therefore, if from it we ſubtract the former triangle, of which the sides about the angle A are b and c, we ſhall have a ſpherical quadrilateral, the baſe of which is A, and the sides perpendicular to that baſe $D - b$ and $D - c$, and the area of the quadrilateral ſhall be expreſſed simply by $B + C - 2D$.

But by the known analogies of Napier, we have in every ſpherical triangle this equation, which will be demonſtrated farther on,

$$\tan.\ \frac{B+C}{2} = \frac{\operatorname{cof.}\ \frac{b-c}{2}}{\operatorname{cof.}\ \frac{b+c}{2}}\ \cot.\ \frac{A}{2}.$$

Therefore, if σ denote the ſurface of the quadrilateral we are conſidering, we ſhall have

$$\tan.\ \frac{\sigma}{2} = \cot.\ \frac{B+C}{2} = \frac{1}{\tan.\ \frac{B+C}{2}};$$ therefore,

$$\tan.\ \frac{\sigma}{2} = \frac{\operatorname{cof.}\ \frac{b+c}{2}}{\operatorname{cof.}\ \frac{b-c}{2}}\ \tan.\ \frac{A}{2};$$

and if we put β and γ for the two ſides of the quadrilateral perpendicular to the baſe A, ſo that $\beta = D - b$ and $\gamma = D - c$ we have in order to determine the area σ this formula

$$\tan.\ \frac{\sigma}{2} = \frac{\sin.\ \frac{\beta+\gamma}{2}}{\operatorname{cof.}\ \frac{\beta-\gamma}{2}}\ \tan.\ \frac{A}{2}.$$

This formula correſponds to the known formula $\sigma = \frac{\beta+\gamma}{2} A$ for a rectilineal quadrilateral of which A is the baſe, β and γ the two vertical ſides, and σ the area; and as the latter is of the greateſt utility in the menſuration of plane ſurfaces bounded by ſtraight lines, ſo the former is equally uſeful in the menſuration of ſpherical ſurfaces bounded by arches of great circles. Thus it may be employed with much advantage to determine the extent of a country, when the latitudes and longitudes of ſeveral points in its circumference are known; for by connecting thoſe points by arches of great circles, we have a ſpherical polygon, the area of which may be eaſily found, by reſolving it into quadrilaterals formed by circles of latitudes and arches of the equator intercepted between thoſe circles.

10. But if it was required to expreſs the area Σ by the three ſides a, b, c of the ſpherical triangle, it is only neceſſary to conſider, that ſince $\Sigma = A + B + C - 2D$ we have

$$\cot.\ \frac{\Sigma}{2} = -\tan.\ \frac{A+B+C}{2}$$

$$= -\frac{\tan.\ \frac{A}{2} + \tan.\ \frac{B+C}{2}}{1 - \tan.\ \frac{A}{2}\ \tan.\ \frac{B+C}{2}}.$$

therefore coſ. A $= \frac{b^2+c^2-a^2}{2bc}$, and hence

$$\text{ſin. A} = \frac{\sqrt{[4b^2c^2-(b^2+c^2-a^2)^2]}}{2bc}.$$

Therefore, if in order to ſimplify, we put

$$d = \sqrt{[4b^2c^2-(b^2+c^2-a^2)^2]}$$

we have ſin. A $= \frac{d}{2bc}$, and hence $\frac{a}{\text{ſin. A}} = \frac{2abc}{d}$ which fraction expreſſes the ratio ſought.

If r be put for the radius of the circle which circumſcribes the triangle, and s for the ſurface, or area of the ſame triangle, we have $\frac{a}{\text{ſin. A}} = \frac{2abc}{d} = 2r = \frac{abc}{2s}$; therefore $r = \frac{abc}{d}$, and and $s = \frac{abc}{4r} = \frac{d}{4}$.

2. By developing the ſquare of $b^2+c^2-a^2$, and reducing, we have

$$d = \sqrt{(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)}$$

a formula in which the three ſides a, b, c are equally concerned, as they ought to be.

But this formula may be rendered more ſimple and more commodious for logarithmic calculation by reſolving it into factors. For

$$4b^2c^2-(b^2+c^2-a^2)^2 = (2bc+b^2+c^2-a^2)\,(2bc-b^2-c^2+a^2),$$

and again reſolving each of theſe factors into two, we have

$$d = \sqrt{[(a+b+c)\,(a-b+c)\,(a+b-c)\,(-a+b+c)]}.$$

Theſe formulas are already known, and I only give them here as an introduction to inveſtigations which are to follow.

3. Since in ſpherical triangles the ſines of the ſides are proportional to the ſines of the angles oppoſite to them, we may be deſirous to inveſtigate the conſtant ratio, and to diſcover if it depends, as in rectilineal triangles, upon the radius of the circumſcribed circle, or the area of the triangle.

Let us denote as before the three ſides of a ſpherical triangle by a, b, c, and the three angles oppoſite to thoſe ſides by A, B, C.

Then by a known theorem

$$\text{coſ. } a = \text{coſ. } b \text{ coſ. } c + \text{ſin. } b \text{ ſin. } c \text{ coſ. A};$$

therefore coſ. A $= \frac{\text{coſ. } a - \text{coſ. } b \text{ coſ. } c}{\text{ſin. } b \text{ ſin. } c}$; and hence

$$\text{ſin. A} = \frac{\sqrt{[\text{ſin.}^2 b \text{ ſin.}^2 c - (\text{coſ. } a - \text{coſ. } b \text{ coſ. } c)^2]}}{\text{ſin. } b \text{ ſin. } c}$$

Let us, to abreviate, put

$$f = \sqrt{[\text{ſin.}^2 b\ \text{ſin.}^2 c - (\text{coſ.}\ a - \text{coſ.}\ b\ \text{coſ.}\ c)^2]}$$

thus we have ſin. A $= \frac{f}{\text{ſin.}\ b\ \text{ſin.}\ c}$, and therefore

$$\frac{\text{ſin.}\ a}{\text{ſin. A}} = \frac{\text{ſin.}\ a\ \text{ſin.}\ b\ \text{ſin.}\ c}{f}$$

an expreſſion for the ratio ſought analogous to that which has been already found for rectilineal triangles (No. 1).

As the quantity f is expreſſed by a ſquare root, and may, therefore, have either the ſign plus, or minus prefixed to it, we remark, that relatively to ſpherical triangles, it ought to be always poſitive; for the ſides and the angles of every triangle being always leſs than two right angles, the ſines are neceſſarily always poſitive.

4. The radical quantity f is alſo ſuſceptible of reductions ſimilar to thoſe of No. 2. For by ſubſtituting inſtead of ſin.^{2} b and ſin.^{2} c their values $1 - \text{coſ.}^2 b$ and $1 - \text{coſ.}^2 c$; and afterwards reducing, we have

$$f = \sqrt{[1 - \text{coſ.}^2 a - \text{coſ.}^2 b - \text{coſ.}^2 c + 2\text{coſ.}\ a\ \text{coſ.}\ b\ \text{coſ.}\ c]}$$

where it appears that the three ſides a, b, c are alike concerned.

In like manner the quantity under the radical ſign may be reſolved into factors, for we have preſently

$$\begin{aligned} &\text{ſin.}^2 b\ \text{ſin.}^2 c - (\text{coſ.}\ a - \text{coſ.}\ b\ \text{coſ.}\ c)^2 \\ &= (\text{ſin.}\ b\ \text{ſin.}\ c + \text{coſ.}\ a - \text{coſ.}\ b\ \text{coſ.}\ c) \\ &\times (\text{ſin.}\ b\ \text{ſin.}\ c - \text{coſ}\ a + \text{coſ.}\ b\ \text{coſ.}\ c) \\ &= [\text{coſ.}\ a - \text{coſ.}\ (b + c)]\ [\text{coſ.}\ (b - c) - \text{coſ.}\ a] \end{aligned}$$

but we have, in general $\text{coſ.}\ a - \text{coſ.}\ h = 2\ \text{ſin.}\ \frac{a+h}{2} \times \text{ſin.}\ \frac{h-a}{2}$

therefore, reſolving thus the two factors, we have

$$f = 2\sqrt{[\text{ſin.}\frac{a+b+c}{2} \times \text{ſin.}\frac{a-b+c}{2} \times \text{ſin.}\frac{a+b-c}{2} \times \text{ſin.}\frac{-a+b+c}{2}]}$$

an expreſſion very convenient for logarithmic calculation.

5. Let us now ſeek the radius of the circle circumſcribing the ſpherical triangle. It is evident that this circle muſt be a ſmall circle of the ſphere, and that it will alſo circumſcribe the rectilineal triangle formed by the chords of the arches a, b, c; but theſe chords being expreſſed by $2\ \text{ſin.}\ \frac{a}{2}$, $2\text{ſin.}\ \frac{b}{2}$, $2\text{ſin.}\ \frac{2}{c}$ it is only neceſſary to ſubſtitute theſe quantitities inſtead of a, b,

c in the expreſſion of the radius r (No. 2), that is to ſay, in $\frac{abc}{d}$.

Let us put R for the radius of the circumſcribed circle, and h for the value which d acquires by ſubſtituting for a, b, c as above; thus we have

$$R = \frac{8\,\text{ſin.}\,\frac{a}{2} \times \text{ſin.}\,\frac{b}{2} \times \text{ſin.}\,\frac{c}{2}}{h}.$$

But we have $h^2 = 32\,(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{b}{2})^2 + 32\,(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{c}{2})^2 + 32\,(\text{ſin.}\,\frac{b}{2}\,\text{ſin.}\,\frac{c}{2})^2 - 16\,(\text{ſin.}\,\frac{a}{2})^4 - 16(\text{sin.}\,\frac{b}{2})^4 - 16(\text{sin.}\,\frac{c}{2})^4$;

and ſubſtituting for $2(\text{sin.}\,\frac{a}{2})^2$, $2(\text{sin.}\,\frac{b}{2})^2$, $2(\text{sin.}\,\frac{c}{2})^2$ their values $1 - \text{coſ.}\,a$, $1 - \text{coſ.}\,b$, $1 - \text{coſ.}\,c$ we have after proper reductions

$$\begin{aligned} h^2 = {} & 12 - 8\,\text{coſ.}\,a - 8\,\text{coſ.}\,b - 8\,\text{coſ.}\,c \\ & + 8\,\text{coſ.}\,a\,\text{coſ.}\,b + 8\,\text{coſ.}\,a\,\text{coſ.}\,c + 8\,\text{coſ.}\,b\,\text{coſ.}\,c \\ & - 4\,\text{coſ.}^2\,a - 4\,\text{coſ.}^2\,b - 4\,\text{coſ.}^2\,c \end{aligned}$$

which expreſſion may be reduced to the following

$$4f^2 + 8(1 - \text{coſ.}\,a)\,(1 - \text{coſ.}\,b)\,(1 - \text{coſ.}\,c)$$

that is to ſay, to

$$4f^2 + 64\,(\text{ſin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2})^2$$

by ſubſtituting the value of f of No. 4. Thus we have

$$R = \frac{4\,\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2}}{\sqrt{[f^2 + 16(\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2})^2]}}.$$

6. Now by conſidering the radius of the ſphere which paſſes through the centre of the ſmall circle circumſcribing the triangle, it is evident that this radius is perpendicular to the plane of the circle, and that it meets the ſurface of the ſphere at a point which is the pole of the ſame circle.

Therefore, putting φ for the arch which meaſures the diſtance of the circle from its pole, we have evidently $R = \text{sin.}\,\varphi$, therefore

Hence we find

$\text{tan.}^2 b + \text{tan.}^2 c - 2\,\text{tan.}\,b\,\text{tan.}\,c\,\text{coſ.}\,A$

$= \text{ſec.}^2 b + \text{ſec.}^2 c - 2\,\text{ſec.}\,b\,\text{ſec.}\,c\,\text{coſ.}\,a.$

But we have evidently

$\text{ſec.}^2 b - \text{tan.}^2 b = 1$, alſo $\text{ſec.}^2 c - \text{tan.}^2 c = 1$,

therefore, the equation becomes

$\text{ſec.}\,b\,\text{ſec.}\,c\,\text{coſ.}\,a = 1 + \text{tan.}\,b\,\text{tan.}\,c\,\text{coſ.}\,A,$

hence, ſubſtituting for ſec. b, ſec. c, tan. b, tan. c, their values $\frac{1}{\text{coſ.}\,b}$, $\frac{1}{\text{coſ.}\,c}$, $\frac{\text{sin.}\,b}{\text{coſ.}\,b}$, $\frac{\text{sin.}\,c}{\text{coſ.}\,c}$, we have the fundamental equation,

$\text{coſ.}\,a = \text{coſ.}\,b\,\text{coſ.}\,c + \text{sin.}\,b\,\text{sin.}\,c\,\text{coſ.}\,A \ldots\ldots\ldots\ldots (A)$

As, by hypotheſis, the only condition ſuppoſed to have place among the four quantities a, b, c, A is that a, b, c are the three ſides of a triangle, and A the angle oppoſite to the ſide a; it follows, that by putting B and C for the angles oppoſite to the ſides b and c, we ſhall have ſimilar equations relatively to thoſe angles, by ſimply changing A into B, or into C, provided that at the ſame time we change a into b, or into c.

15. Now if we deduce the value of coſ. A from the preceding equations, and thence that of sin. A we ſhall have, as has been already found in No. 3,

$$\frac{\text{sin.}\,a}{\text{sin.}\,A} = \frac{\text{sin.}\,a\,\text{sin.}\,b\,\text{sin.}\,c}{f}$$

where the quantity f is a function of a, b, c into which theſe three quantities enter ſimilarly, ſo that it remains the ſame, whatever permutations be made among them. Therefore, in changing a into b, and b into a, the ſecond member of the equation will be the ſame as before; hence we have this other equation,

$$\frac{\text{sin.}\,a}{\text{sin.}\,A} = \frac{\text{sin.}\,b}{\text{sin.}\,B} \ldots\ldots\ldots\ldots (B)$$

This is what is called the common analogy of the sines, and it is obvious, that by changing a and c into A and C, we have in like manner $\frac{\text{sin.}\,a}{\text{sin.}\,A} = \frac{\text{sin.}\,c}{\text{sin.}\,C}$.

16. Let us reſume the equation (A) of No. 14, $\text{coſ.}\,a = \text{coſ.}\,b\,\text{coſ.}\,c + \text{sin.}\,b\,\text{sin.}\,c\,\text{coſ.}\,A$; by changing a into c, and A into C, we have in like manner $\text{coſ.}\,c = \text{coſ.}\,a\,\text{coſ.}\,b + \text{sin.}\,a\,\text{sin.}\,b\,\text{coſ.}\,C$; ſubſtituting this value of coſ. c in the firſt equation it becomes $\text{coſ.}\,a = \text{coſ.}\,a\,\text{coſ.}^2 b + \text{sin.}\,a\,\text{sin.}\,b\,\text{coſ.}\,b\,\text{coſ.}\,C$

$$\frac{2\sin.\frac{a}{2}\sin.\frac{b}{2}\sin.\frac{c}{2}}{3\tan.\varphi}$$

and again, by ſubſtituting for tan. φ its value already found (No. 6), we have $\frac{f}{6}$ for the ſolidity of the pyramid.

8. It remains for us to conſider the area of the ſpherical triangle itſelf formed by the arcs a, b, c. There is a beautiful and known theorem, according to which, the area of a ſpherical triangle is to the whole ſurface of the ſphere, as the exceſs of the three angles of the triangle above two right angles to eight right angles. This theorem is commonly attributed to Albert Girard, who gave it in ſubſtance, in a work entitled *Invention Nouvelle en Algebre*, printed at Amſterdam in 1629; but as the proof which he gives is not rigorous, and cannot even be conſidered as an induction, the theorem ought rather to be attributed to Cavallerius, who gave it in the *Directericum generale uranometricum*, printed at Bologna in 1632, with the elegant demonſtration firſt publiſhed by Wallis, and ſince inſerted in moſt works upon trigonometry.

Let Σ denote the exceſs of the three angles of the triangle above two right angles, then, retaining the notation hitherto employed, and putting D for a right angle,

$$\Sigma = A + B + C - 2D.$$

Thus the area of the triangle of which the ſides are a, b, c and the angles A, B, C ſhall be the $\frac{\Sigma}{8D}$ part of the whole ſurface of the ſphere; and if this ſurface be conſidered as equal to 8D, we may take Σ for the area of the triangle.

9. If we ſuppoſe the ſides b and c, which contain the angle A, to be produced till each of them be a quadrant, the angles B, C will then become right angles, and the ſide a will become equal to the oppoſite angle A; then the area of the iſoſceles right angled triangle becomes A; therefore, if from it we ſubtract the former triangle, of which the ſides about the angle A are b and c, we ſhall have a ſpherical quadrilateral, the baſe of which is A, and the ſides perpendicular to that baſe $D - b$ and $D - c$, and the area of the quadrilateral ſhall be expreſſed simply by $B + C - 2D$.

But by the known analogies of Napier, we have in every ſpherical triangle this equation, which will be demonſtrated farther on,

metry; for as a ſpherical triangle conſiſts of but ſix parts, the three ſides, and the three angles, and as three of theſe are ſufficient to determine the triangle, it is evident that the moſt ſimple relations can only take place among four of thoſe parts: But all the different combinations that can be made of ſix things, taken four by four, are the following:

1. Between the three ſides and an angle: this relation is given by the equation (A).

2. Between two ſides and two angles, which may be reſpectively oppoſite to the two ſides, or the one oppoſite, and the other adjacent to the ſame ſide, making two caſes. The relation between two ſides, and the two opposite angles, is given by equation (B).

3. Between two sides and two angles, of which one is opposite, and the other adjacent to the ſame side: this relation is contained in equation (C)

4. Between the three angles and a side: this relation is given by equation (D).

20. If A be ſuppoſed a right angle, the preceding equations become more simple, and give theſe other equations:

$$\text{coſ. } a = \text{coſ. } b \text{ coſ. } c$$

$$\text{sin. } a = \frac{\text{sin. } b}{\text{sin. B}}$$

$$\text{cot. } a = \text{cot. } b \text{ coſ. C}$$

$$\text{coſ. } a = \text{cot. B cot. C},$$

and if C be ſuppoſed a right angle, the equations (C) and (D) give theſe other two:

$$\text{Cot. A} = \text{cot. } a \text{ sin. } b,$$

$$\text{Cot. A} = \text{sin. B coſ. } a.$$

Theſe six equations give immediately the ſolution of all caſes of right-angled ſpherical triangles; and as their form readily admits of the application of logarithms, they are commonly employed in Trigonometry, by reſolving all triangles into right-angled triangles, by letting fall a perpendicular. But all the caſes may be alſo reſolved by the four general equations, which may be reduced into factors by means of transformations, which we now proceed to explain.

21. The equation (A) between the three sides a, b, c, and an angle A opposite to the side a may ſerve to determine: 1st, A by a, b, c; 2d, a by, b, c, and A; 3d, b by a, c, and A.

1st, To determine A by a, b, c, we have

$$\text{Coſ. A} = \frac{\text{coſ. } a - \text{coſ. } b \text{ coſ. } c}{\text{sin. } b \text{ sin. } c}$$

Hence

$$\frac{2\sin.\frac{a}{2}\ \sin.\frac{b}{2}\ \sin.\frac{c}{2}}{3\tan.\ \varphi}$$

and again, by ſubſtituting for tan. φ its value already found (No. 6), we have $\frac{\mathcal{L}}{6}$ for the ſolidity of the pyramid.

8. It remains for us to conſider the area of the ſpherical triangle itſelf formed by the arcs a, b, c. There is a beautiful and known theorem, according to which, the area of a ſpherical triangle is to the whole ſurface of the ſphere, as the exceſs of the three angles of the triangle above two right angles to eight right angles. This theorem is commonly attributed to Albert Girard, who gave it in ſubſtance, in a work entitled *Invention Nouvelle en Algebre*, printed at Amſterdam in 1629; but as the proof which he gives is not rigorous, and cannot even be conſidered as an induction, the theorem ought rather to be attributed to Cavallerius, who gave it in the *Directericum generale uranometricum*, printed at Bologna in 1632, with the elegant demonſtration firſt publiſhed by Wallis, and since inſerted in moſt works upon trigonometry.

Let Σ denote the exceſs of the three angles of the triangle above two right angles, then, retaining the notation hitherto employed, and putting D for a right angle,

$$\Sigma = A + B + C - 2D.$$

Thus the area of the triangle of which the sides are a, b, c and the angles A, B, C ſhall be the $\frac{\Sigma}{8D}$ part of the whole ſurface of the ſphere; and if this ſurface be conſidered as equal to 8D, we may take Σ for the area of the triangle.

9. If we ſuppoſe the sides b and c, which contain the angle A, to be produced till each of them be a quadrant, the angles B, C will then become right angles, and the side a will become equal to the oppoſite angle A; then the area of the iſoſceles right angled triangle becomes A; therefore, if from it we ſubtract the former triangle, of which the sides about the angle A are b and c, we ſhall have a ſpherical quadrilateral, the baſe of which is A, and the sides perpendicular to that baſe $D - b$ and $D - c$, and the area of the quadrilateral ſhall be expreſſed simply by $B + C - 2D$.

But by the known analogies of Napier, we have in every ſpherical triangle this equation, which will be demonſtrated farther on,

$$\text{tan.}\,\frac{B+C}{2} = \frac{\text{coſ.}\,\frac{b-c}{2}}{\text{coſ.}\,\frac{b+c}{2}}\,\text{cot.}\,\frac{A}{2}.$$

Therefore, if σ denote the ſurface of the quadrilateral we are conſidering, we ſhall have

$$\text{tan.}\,\frac{\sigma}{2} = \text{cot.}\,\frac{B+C}{2} = \frac{1}{\text{tan.}\,\frac{B+C}{2}};\ \text{therefore,}$$

$$\text{tan.}\,\frac{\sigma}{2} = \frac{\text{coſ.}\,\frac{b+c}{2}}{\text{coſ.}\,\frac{b-c}{2}}\,\text{tan.}\,\frac{A}{2};$$

and if we put β and γ for the two ſides of the quadrilateral perpendicular to the baſe A, ſo that $\beta = D - b$ and $\gamma = D - c$ we have in order to determine the area σ this formula

$$\text{tan.}\,\frac{\sigma}{2} = \frac{\text{ſin.}\,\frac{\beta+\gamma}{2}}{\text{coſ.}\,\frac{\beta-\gamma}{2}}\,\text{tan.}\,\frac{A}{2}.$$

This formula correſponds to the known formula $\sigma = \frac{\beta+\gamma}{2}A$ for a rectilineal quadrilateral of which A is the baſe, β and γ the two vertical ſides, and σ the area; and as the latter is of the greateſt utility in the menſuration of plane ſurfaces bounded by ſtraight lines, ſo the former is equally uſeful in the menſuration of ſpherical ſurfaces bounded by arches of great circles. Thus it may be employed with much advantage to determine the extent of a country, when the latitudes and longitudes of ſeveral points in its circumference are known; for by connecting thoſe points by arches of great circles, we have a ſpherical polygon, the area of which may be eaſily found, by reſolving it into quadrilaterals formed by circles of latitudes and arches of the equator intercepted between thoſe circles.

10. But if it was required to expreſs the area Σ by the three ſides a, b, c of the ſpherical triangle, it is only neceſſary to conſider, that ſince $\Sigma = A + B + C - 2D$ we have

$$\text{cot.}\,\frac{\Sigma}{2} = -\,\text{tan.}\,\frac{A+B+C}{2}$$

$$= -\,\frac{\text{tan.}\,\frac{A}{2} + \text{tan.}\,\frac{B+C}{2}}{1 - \text{tan.}\,\frac{A}{2}\,\text{tan.}\,\frac{B+C}{2}}.$$

If we ſubſtitute, in place of tan. $\frac{B+C}{2}$ its value found above (preceding No.), we have

$$\text{cot.}\ \frac{\Sigma}{2} = -\ \frac{\text{tan.}\ \frac{A}{2}\ \text{coſ.}\ \frac{b+c}{2} + \text{cot.}\ \frac{A}{2}\ \text{coſ.}\ \frac{b-c}{2}}{\text{coſ.}\ \frac{b+c}{2} - \text{coſ.}\ \frac{b-c}{2}},$$

a formula, which is eaſily transformed to this other

$$\text{cot.}\ \frac{\Sigma}{2} = \frac{\text{coſ.}\ \frac{b}{2}\ \text{coſ.}\ \frac{c}{2} + \text{ſin.}\ \frac{b}{2}\ \text{ſin.}\ \frac{c}{2}\ \text{coſ.}\ A}{\text{ſin.}\ \frac{b}{2}\ \text{ſin.}\ \frac{c}{2}\ \text{ſin.}\ A}.$$

If now we ſubſtitute in this formula the values ſin. A, and coſ. A of No. 3, we ſhall have, by dividing the numerator and denominator by ſin. $\frac{b}{2}$ ſin. $\frac{c}{2}$

$$\text{cot.}\ \frac{\Sigma}{2} = \frac{4(\text{coſ.}\ \frac{b}{2}\ \text{coſ.}\ \frac{c}{2})^2 + \text{coſ.}\ a - \text{coſ.}\ b\ \text{coſ.}\ c}{f};$$

But $2(\text{coſ.}\ \frac{b}{2})^2 = 1 + \text{coſ.}\ b$, and $2(\text{coſ.}\ \frac{c}{2})^2 = 1 + \text{coſ.}\ c$, therefore, making the neceſſary ſubſtitutions and inverting the fraction,

$$\text{tan.}\ \frac{\Sigma}{2} = \frac{f}{1 + \text{coſ.}\ a + \text{coſ.}\ b + \text{coſ.}\ c}$$

a formula of the moſt ſimple form to determine the area of a ſpherical triangle by means of its three ſides a, b, c.

11. We have ſeen (No. 7) that $\frac{f}{6}$ is the ſolidity of the triangular pyramid formed by the three radii of the ſphere which terminate in the angles of a ſpherical triangle. Let us now conſider a triangular pyramid formed by the ſame radii produced at pleaſure in ſuch a manner that they become p, q, r; and that a, b, c are the arches, or angles contained by thoſe ſtraight lines. To have the ſolidity of this pyramid, it may be conſidered as lying on one of its faces; for inſtance, that which has for its ſides the lines p, q, and to have a perpendicular P drawn from the other ſide r upon the plane of that face. It is immediately obvious, that if a is the angle contained by p and q, the area of the face which we have conſidered as the baſe of the

pyramid will be $\frac{pq\ \text{ſin.}\ a}{2}$, therefore, the ſolidity of the pyramid will be $\frac{Ppq\ \text{ſin.}\ a}{6}$. But if we call θ the angle which the line r makes with the plane paſſing through p and q, it is evident that we have $P = r$ ſin. θ, therefore, the ſolidity required will be $\frac{pqr\ \text{ſin.}\ a\ \text{ſin.}\ \theta}{6}$.

The angle θ is no other than the arch let fall perpendicularly from the angle A of the ſpherical triangle upon the oppoſite ſide a; we may, therefore, expreſs ſin. θ by the ſines or coſines of a, b, c the ſides of the triangle; but for our purpoſe it is ſufficient to conſider that this value, as well as that of ſin. a, being independent of the lines p, q, r, if we make $p = 1$, $q = 1$, $r = 1$ we ſhall have the caſe of the particular pyramid we propoſe to conſider; and of which the ſolidity is $\frac{f}{6}$. Hence it follows, that we have ſin. a ſin. $\theta = f$, and therefore $\frac{pqrf}{6}$ for the ſolidity of the triangular pyramid in which the three ſides, or edges that form any one of the ſolid angles are p, q, r, and the angles contained by thoſe three ſides are A, B, C.

This expreſſion, for the ſolidity of any triangular pyramid, by means of its three ſides and contained angles, is evidently very ſimple and commodious for calculation, more eſpecially if we employ the value of f, as expreſſed by factors, in No. 4; and it may be very uſeful to determine the ſolid contents of all bodies bounded by planes, ſeeing that they may always be reſolved into triangular pyramids, as a polygon is reſolved into triangles.

12. We may add, that ſince we have found ſin. a ſin. $\theta = f$, we ſhall have ſin. $\theta = \frac{f}{\text{ſin.}\ a}$. We may thus determine by this formula the perpendicular θ in a ſpherical triangle, of which a is the baſe, and b, c the two ſides.

13. I have reſolved the preceding problems, only to ſhew the origin and uſe of ſome remarkable formulas, and more eſpecially of that function which I have denoted by f, and which deſerves in a particular manner the attention of analyſts, on account of its different applications. I now proceed to general conſiderations relating to ſpherical trigonometry, viewed analytically.

Analytical reſolutions of ſpherical triangles were at firſt only

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therefore $\text{coſ. A} = \frac{b^2+c^2-a^2}{2bc}$, and hence

$$\text{ſin. A} = \frac{\sqrt{[4b^2c^2-(b^2+c^2-a^2)^2]}}{2bc}.$$

Therefore, if in order to ſimplify, we put

$$d = \sqrt{[4b^2c^2-(b^2+c^2-a^2)^2]}$$

we have $\text{ſin. A} = \frac{d}{2bc}$, and hence $\frac{a}{\text{ſin. A}} = \frac{2abc}{d}$ which fraction expreſſes the ratio ſought.

If r be put for the radius of the circle which circumſcribes the triangle, and s for the ſurface, or area of the ſame triangle, we have $\frac{a}{\text{ſin. A}} = \frac{2abc}{d} = 2r = \frac{abc}{2s}$; therefore $r = \frac{abc}{d}$, and and $s = \frac{abc}{4r} = \frac{d}{4}$.

2. By developing the ſquare of $b^2+c^2-a^2$, and reducing, we have

$$d = \sqrt{(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)}$$

a formula in which the three ſides a, b, c are equally concerned, as they ought to be.

But this formula may be rendered more ſimple and more commodious for logarithmic calculation by reſolving it into factors. For

$$4b^2c^2-(b^2+c^2-a^2)^2 = (2bc+b^2+c^2-a^2)\,(2bc-b^2-c^2+a^2),$$

and again reſolving each of theſe factors into two, we have

$$d = \sqrt{[(a+b+c)\,(a-b+c)\,(a+b-c)\,(-a+b+c)]}.$$

Theſe formulas are already known, and I only give them here as an introduction to inveſtigations which are to follow.

3. Since in ſpherical triangles the ſines of the ſides are proportional to the ſines of the angles oppoſite to them, we may be deſirous to inveſtigate the conſtant ratio, and to diſcover if it depends, as in rectilineal triangles, upon the radius of the circumſcribed circle, or the area of the triangle.

Let us denote as before the three ſides of a ſpherical triangle by a, b, c, and the three angles oppoſite to thoſe ſides by A, B, C.

Then by a known theorem

$$\text{coſ. } a = \text{coſ. } b \text{ coſ. } c + \text{ſin. } b \text{ ſin. } c \text{ coſ. A};$$

therefore $\text{coſ. A} = \frac{\text{coſ. } a - \text{coſ. } b \text{ coſ. } c}{\text{ſin. } b \text{ ſin. } c}$; and hence

$$\text{ſin. A} = \frac{\sqrt{[\text{ſin.}^2 b \text{ ſin.}^2 c-(\text{coſ. } a-\text{coſ. } b \text{ coſ. } c)^2]}}{\text{ſin. } b \text{ ſin. } c}$$

Let us, to abreviate, put

$$f = \sqrt{[\text{ſin.}^2 b \,\text{ſin.}^2 c - (\text{coſ.}\, a - \text{coſ.}\, b\, \text{coſ.}\, c)^2]}$$

thus we have $\text{ſin. A} = \frac{f}{\text{ſin.}\, b\, \text{ſin.}\, c}$, and therefore

$$\frac{\text{ſin.}\, a}{\text{ſin. A}} = \frac{\text{ſin.}\, a\, \text{ſin.}\, b\, \text{ſin.}\, c}{f}$$

an expreſſion for the ratio ſought analogous to that which has been already found for rectilineal triangles (No. 1).

As the quantity f is expreſſed by a ſquare root, and may, therefore, have either the ſign plus, or minus prefixed to it, we remark, that relatively to ſpherical triangles, it ought to be always poſitive; for the ſides and the angles of every triangle being always leſs than two right angles, the ſines are neceſſarily always poſitive.

4. The radical quantity f is alſo ſuſceptible of reductions ſimilar to thoſe of No. 2. For by ſubſtituting inſtead of $\text{ſin.}^2 b$ and $\text{ſin.}^2 c$ their values $1 - \text{coſ.}^2 b$ and $1 - \text{coſ.}^2 c$; and afterwards reducing, we have

$$f = \sqrt{[1 - \text{coſ.}^2 a - \text{coſ.}^2 b - \text{coſ.}^2 c + 2\text{coſ.}\, a\, \text{coſ.}\, b\, \text{coſ.}\, c]}$$

where it appears that the three ſides a, b, c are alike concerned.

In like manner the quantity under the radical ſign may be reſolved into factors, for we have preſently

$$\begin{aligned} &\text{ſin.}^2 b\, \text{ſin.}^2 c - (\text{coſ.}\, a - \text{coſ.}\, b\, \text{coſ.}\, c)^2 \\ &= (\text{ſin.}\, b\, \text{ſin.}\, c + \text{coſ.}\, a - \text{coſ.}\, b\, \text{coſ.}\, c) \\ &\quad \times (\text{ſin.}\, b\, \text{ſin.}\, c - \text{coſ}\, a + \text{coſ.}\, b\, \text{coſ.}\, c) \\ &= [\text{coſ.}\, a - \text{coſ.}\, (b + c)]\, [\text{coſ.}\, (b - c) - \text{coſ.}\, a] \end{aligned}$$

but we have, in general $\text{coſ.}\, a - \text{coſ.}\, h = 2\, \text{ſin.}\, \frac{a+h}{2} \times \text{ſin.}\, \frac{h-a}{2}$

therefore, reſolving thus the two factors, we have

$$f = 2\sqrt{[\text{ſin.}\frac{a+b+c}{2} \times \text{ſin.}\frac{a-b+c}{2} \times \text{ſin.}\frac{a+b-c}{2} \times \text{ſin.}\frac{-a+b+c}{2}]}$$

an expreſſion very convenient for logarithmic calculation.

5. Let us now ſeek the radius of the circle circumſcribing the ſpherical triangle. It is evident that this circle muſt be a ſmall circle of the ſphere, and that it will alſo circumſcribe the rectilineal triangle formed by the chords of the arches a, b, c; but theſe chords being expreſſed by $2\, \text{ſin.}\, \frac{a}{2}$, $2\text{ſin.}\, \frac{b}{2}$, $2\text{ſin.}\, \frac{2}{c}$ it is only neceſſary to ſubſtitute theſe quantitities inſtead of a, b,

c in the expreſſion of the radius r (No. 2), that is to ſay, in $\frac{abc}{d}$.

Let us put R for the radius of the circumſcribed circle, and h for the value which d acquires by ſubſtituting for a, b, c as above; thus we have

$$R = \frac{8\,\text{ſin.}\,\frac{a}{2} \times \text{ſin.}\,\frac{b}{2} \times \text{ſin.}\,\frac{c}{2}}{h}.$$

But we have $h^2 = 32\,(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{b}{2})^2 + 32\,(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{c}{2})^2 + 32\,(\text{ſin.}\,\frac{b}{2}\,\text{ſin.}\,\frac{c}{2})^2 - 16\,(\text{ſin.}\,\frac{a}{2})^4 - 16(\text{sin.}\,\frac{b}{2})^4 - 16(\text{sin.}\,\frac{c}{2})^4$;

and ſubſtituting for $2(\text{sin.}\,\frac{a}{2})^2$, $2(\text{sin.}\,\frac{b}{2})^2$, $2(\text{sin.}\,\frac{c}{2})^2$ their values $1 - \text{coſ.}\,a$, $1 - \text{coſ.}\,b$, $1 - \text{coſ.}\,c$ we have after proper reductions

$$\begin{aligned} h^2 = {} & 12 - 8\,\text{coſ.}\,a - 8\,\text{coſ.}\,b - 8\,\text{coſ.}\,c \\ & + 8\,\text{coſ.}\,a\,\text{coſ.}\,b + 8\,\text{coſ.}\,a\,\text{coſ.}\,c + 8\,\text{coſ.}\,b\,\text{coſ.}\,c \\ & - 4\,\text{coſ.}^2\,a - 4\,\text{coſ.}^2\,b - 4\,\text{coſ.}^2\,c \end{aligned}$$

which expreſſion may be reduced to the following

$$4f^2 + 8(1 - \text{coſ.}\,a)\,(1 - \text{coſ.}\,b)\,(1 - \text{coſ.}\,c)$$

that is to ſay, to

$$4f^2 + 64\,(\text{ſin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2})^2$$

by ſubſtituting the value of f of No. 4. Thus we have

$$R = \frac{4\,\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2}}{\sqrt{[f^2 + 16(\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2})^2]}}.$$

6. Now by conſidering the radius of the ſphere which paſſes through the centre of the ſmall circle circumſcribing the triangle, it is evident that this radius is perpendicular to the plane of the circle, and that it meets the ſurface of the ſphere at a point which is the pole of the ſame circle.

Therefore, putting φ for the arch which meaſures the diſtance of the circle from its pole, we have evidently $R = \text{sin.}\,\varphi$, therefore

sin.

This, however, would be to demonſtrate that which is ſimple by that which is compound, and, therefore, of no uſe except as an exerciſe in calculation; we ſhall only remark that equation (D) of No. 17, gives directly this other equation, coſ. A $=$ ſin. B ſin. C $-$ coſ. B coſ. C; or coſ. A $= -$ coſ. (B $-$ C), and, therefore, A $= 2\,\mathrm{D} - \mathrm{B} - \mathrm{C}$, that is, $\mathrm{A} + \mathrm{B} + \mathrm{C} = 2\mathrm{D}$, where D denotes a right angle, which is a known property of rectilineal triangles.

27. Now if the radius of the ſphere, inſtead of being infinitely great, is only very great, the ſpherical triangle will only be nearly rectilineal; and here, as the angles a, b, c, which correſpond to the ſides become very ſmall, the common trigonometrical tables are not ſufficiently preciſe for the calculation of the ſides and angles. In this caſe, therefore, ſpherical triangles may be treated with advantage, as rectilineal triangles, by taking into account the ſmall correction that reſults from their difference.

This caſe always occurs in the calculation of triangles formed upon the earth's ſurface for the meaſurement of an arch of the meridian; on theſe triangles the quantities β, γ, are the length of the arches, and r is the earth's radius.

To determine the correction we reſume equation (A) of No. 14, which ſerves as the baſis of ſpherical trigonometry, and which gives

$$\text{Coſ. A} = \frac{\text{coſ. } a - \text{coſ. } b \text{ coſ. } c}{\text{ſin. } b \text{ ſin. } c}.$$

Making in the ſecond member of this equation the ſubſtitutions indicated in laſt number, and ſtopping at the terms divided by r^4, we have

$$\text{Coſ. A} = \frac{\frac{\beta^2+\gamma^2-\alpha^2}{2r^2} + \frac{\alpha^4-\beta^4-\gamma^4}{2\cdot3\cdot4\,r^4} - \frac{\beta^2\gamma^2}{4r^4}}{\frac{\beta\gamma}{r^2}\left(1 - \frac{\beta^2+\gamma^2}{2\cdot3\gamma^2}\right)},$$

multiplying the numerator and denominator of the fraction by r^2, and ſubſtituting the factor $1 + \frac{\beta^2+\gamma^2}{2\cdot3r^2}$ inſtead of the diviſor $1 - \frac{\beta^2+\gamma^2}{2\cdot3r^2}$, we have, neglecting ſuch terms as are divided by powers above r^2

$$\text{Coſ. A} = \frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma} + \frac{\alpha^4-\beta^4-\gamma^4}{2\cdot3\cdot4\beta\gamma\,r^2} - \frac{\beta^2\gamma^2}{4\beta\gamma\,r^2}$$
$$+ \frac{(\beta^2+\gamma^2-\alpha^2)(\beta^2+\gamma^2)}{3\cdot4\beta\gamma\,r^2},$$

$=$

$$\frac{2\sin.\frac{a}{2}\sin.\frac{b}{2}\sin.\frac{c}{2}}{3\tan.\varphi}$$

and again, by ſubſtituting for tan. φ its value already found (No. 6), we have $\frac{L}{6}$ for the ſolidity of the pyramid.

8. It remains for us to conſider the area of the ſpherical triangle itſelf formed by the arcs a, b, c. There is a beautiful and known theorem, according to which, the area of a ſpherical triangle is to the whole ſurface of the ſphere, as the exceſs of the three angles of the triangle above two right angles to eight right angles. This theorem is commonly attributed to Albert Girard, who gave it in ſubſtance, in a work entitled *Invention Nouvelle en Algebre*, printed at Amſterdam in 1629; but as the proof which he gives is not rigorous, and cannot even be conſidered as an induction, the theorem ought rather to be attributed to Cavallerius, who gave it in the *Directericum generale uranometricum*, printed at Bologna in 1632, with the elegant demonſtration firſt publiſhed by Wallis, and since inſerted in moſt works upon trigonometry.

Let Σ denote the exceſs of the three angles of the triangle above two right angles, then, retaining the notation hitherto employed, and putting D for a right angle,

$$\Sigma = A + B + C - 2D.$$

Thus the area of the triangle of which the sides are a, b, c and the angles A, B, C ſhall be the $\frac{\Sigma}{8D}$ part of the whole ſurface of the ſphere; and if this ſurface be conſidered as equal to 8D, we may take Σ for the area of the triangle.

9. If we ſuppoſe the sides b and c, which contain the angle A, to be produced till each of them be a quadrant, the angles B, C will then become right angles, and the side a will become equal to the oppoſite angle A; then the area of the iſoſceles right angled triangle becomes A; therefore, if from it we ſubtract the former triangle, of which the sides about the angle A are b and c, we ſhall have a ſpherical quadrilateral, the baſe of which is A, and the sides perpendicular to that baſe $D - b$ and $D - c$, and the area of the quadrilateral ſhall be expreſſed simply by $B + C - 2D$.

But by the known analogies of Napier, we have in every ſpherical triangle this equation, which will be demonſtrated farther on,

tan.

$$\tan.\ \frac{B+C}{2}=\frac{\text{coſ.}\ \frac{b-c}{2}}{\text{coſ.}\ \frac{b+c}{2}}\cot.\ \frac{A}{2}.$$

Therefore, if σ denote the ſurface of the quadrilateral we are conſidering, we ſhall have

$$\tan.\ \frac{\sigma}{2}=\cot.\ \frac{B+C}{2}=\frac{1}{\tan.\ \frac{B+C}{2}};\ \text{therefore,}$$

$$\tan.\ \frac{\sigma}{2}=\frac{\text{coſ.}\ \frac{b+c}{2}}{\text{coſ.}\ \frac{b-c}{2}}\tan.\ \frac{A}{2};$$

and if we put β and γ for the two ſides of the quadrilateral perpendicular to the baſe A, ſo that $\beta=D-b$ and $\gamma=D-c$ we have in order to determine the area σ this formula

$$\tan.\ \frac{\sigma}{2}=\frac{\sin.\ \frac{\beta+\gamma}{2}}{\text{coſ.}\ \frac{\beta-\gamma}{2}}\tan.\ \frac{A}{2}.$$

This formula correſponds to the known formula $\sigma=\frac{\beta+\gamma}{2}A$ for a rectilineal quadrilateral of which A is the baſe, β and γ the two vertical ſides, and σ the area; and as the latter is of the greateſt utility in the menſuration of plane ſurfaces bounded by ſtraight lines, ſo the former is equally uſeful in the menſuration of ſpherical ſurfaces bounded by arches of great circles. Thus it may be employed with much advantage to determine the extent of a country, when the latitudes and longitudes of ſeveral points in its circumference are known; for by connecting thoſe points by arches of great circles, we have a ſpherical polygon, the area of which may be eaſily found, by reſolving it into quadrilaterals formed by circles of latitudes and arches of the equator intercepted between thoſe circles.

10. But if it was required to expreſs the area Σ by the three ſides a, b, c of the ſpherical triangle, it is only neceſſary to conſider, that ſince $\Sigma=A+B+C-2D$ we have

$$\cot.\ \frac{\Sigma}{2}=-\tan.\ \frac{A+B+C}{2}$$

$$=-\frac{\tan.\ \frac{A}{2}+\tan.\ \frac{B+C}{2}}{1-\tan.\ \frac{A}{2}\tan.\ \frac{B+C}{2}}.$$

If we ſubſtitute, in place of tan. $\frac{B+C}{2}$ its value found above (preceding No.), we have

$$\cot.\frac{\Sigma}{2} = -\frac{\tan.\frac{A}{2}\,\text{coſ.}\,\frac{b+c}{2} + \cot.\frac{A}{2}\,\text{coſ.}\,\frac{b-c}{2}}{\text{coſ.}\,\frac{b+c}{2} - \text{coſ.}\,\frac{b-c}{2}},$$

a formula, which is eaſily transformed to this other

$$\cot.\frac{\Sigma}{2} = \frac{\text{coſ.}\,\frac{b}{2}\,\text{coſ.}\,\frac{c}{2} + \text{ſin.}\,\frac{b}{2}\,\text{ſin.}\,\frac{c}{2}\,\text{coſ. A}}{\text{ſin.}\,\frac{b}{2}\,\text{ſin.}\,\frac{c}{2}\,\text{ſin. A}}.$$

If now we ſubſtitute in this formula the values ſin. A, and coſ. A of No. 3, we ſhall have, by dividing the numerator and denominator by ſin. $\frac{b}{2}$ ſin. $\frac{c}{2}$

$$\cot.\frac{\Sigma}{2} = \frac{4(\text{coſ.}\,\frac{b}{2}\,\text{coſ.}\,\frac{c}{2})^2 + \text{coſ.}\,a - \text{coſ.}\,b\,\text{coſ.}\,c}{f};$$

But $2(\text{coſ.}\,\frac{b}{2})^2 = 1 + \text{coſ.}\,b$, and $2(\text{coſ.}\,\frac{c}{2})^2 = 1 + \text{coſ.}\,c$, therefore, making the neceſſary ſubſtitutions and inverting the fraction,

$$\tan.\frac{\Sigma}{2} = \frac{f}{1 + \text{coſ.}\,a + \text{coſ.}\,b + \text{coſ.}\,c}$$

a formula of the moſt ſimple form to determine the area of a ſpherical triangle by means of its three ſides a, b, c.

11. We have ſeen (No. 7) that $\frac{f}{6}$ is the ſolidity of the triangular pyramid formed by the three radii of the ſphere which terminate in the angles of a ſpherical triangle. Let us now conſider a triangular pyramid formed by the ſame radii produced at pleaſure in ſuch a manner that they become p, q, r; and that a, b, c are the arches, or angles contained by thoſe ſtraight lines. To have the ſolidity of this pyramid, it may be conſidered as lying on one of its faces; for inſtance, that which has for its ſides the lines p, q, and to have a perpendicular P drawn from the other ſide r upon the plane of that face. It is immediately obvious, that if a is the angle contained by p and q, the area of the face which we have conſidered as the baſe of the

pyramid will be $\frac{pq\ \text{ſin.}\ a}{2}$, therefore, the ſolidity of the pyramid will be $\frac{Ppq\ \text{ſin.}\ a}{6}$. But if we call θ the angle which the line r makes with the plane paſſing through p and q, it is evident that we have $P = r\ \text{ſin.}\ \theta$, therefore, the ſolidity required will be $\frac{pqr\ \text{ſin.}\ a\ \text{ſin.}\ \theta}{6}$.

The angle θ is no other than the arch let fall perpendicularly from the angle A of the ſpherical triangle upon the oppoſite ſide a; we may, therefore, expreſs ſin. θ by the ſines or coſines of a, b, c the ſides of the triangle; but for our purpoſe it is ſufficient to conſider that this value, as well as that of ſin. a, being independent of the lines p, q, r, if we make $p = 1, q = 1$, $r = 1$ we ſhall have the caſe of the particular pyramid we propoſe to conſider; and of which the ſolidity is $\frac{f}{6}$. Hence it follows, that we have ſin. a ſin. $\theta = f$, and therefore $\frac{pqrf}{6}$ for the ſolidity of the triangular pyramid in which the three ſides, or edges that form any one of the ſolid angles are p, q, r, and the angles contained by thoſe three ſides are A, B, C.

This expreſſion, for the ſolidity of any triangular pyramid, by means of its three ſides and contained angles, is evidently very ſimple and commodious for calculation, more eſpecially if we employ the value of f, as expreſſed by factors, in No. 4; and it may be very uſeful to determine the ſolid contents of all bodies bounded by planes, ſeeing that they may always be reſolved into triangular pyramids, as a polygon is reſolved into triangles.

12. We may add, that ſince we have found ſin. a ſin. $\theta = f$, we ſhall have ſin. $\theta = \frac{f}{\text{ſin.}\ a}$. We may thus determine by this formula the perpendicular θ in a ſpherical triangle, of which a is the baſe, and b, c the two ſides.

13. I have reſolved the preceding problems, only to ſhew the origin and uſe of ſome remarkable formulas, and more eſpecially of that function which I have denoted by f, and which deſerves in a particular manner the attention of analyſts, on account of its different applications. I now proceed to general conſiderations relating to ſpherical trigonometry, viewed analytically.

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$$\frac{2\sin.\frac{a}{2}\,\sin.\frac{b}{2}\,\sin.\frac{c}{2}}{3\tan.\,\varphi}$$

and again, by ſubſtituting for tan. φ its value already found (No. 6), we have $\frac{f}{6}$ for the ſolidity of the pyramid.

8. It remains for us to conſider the area of the ſpherical triangle itſelf formed by the arcs a, b, c. There is a beautiful and known theorem, according to which, the area of a ſpherical triangle is to the whole ſurface of the ſphere, as the exceſs of the three angles of the triangle above two right angles to eight right angles. This theorem is commonly attributed to Albert Girard, who gave it in ſubſtance, in a work entitled *Invention Nouvelle en Algebre*, printed at Amſterdam in 1629; but as the proof which he gives is not rigorous, and cannot even be conſidered as an induction, the theorem ought rather to be attributed to Cavallerius, who gave it in the *Directericum generale uranometricum*, printed at Bologna in 1632, with the elegant demonſtration firſt publiſhed by Wallis, and ſince inſerted in moſt works upon trigonometry.

Let Σ denote the exceſs of the three angles of the triangle above two right angles, then, retaining the notation hitherto employed, and putting D for a right angle,

$$\Sigma = A + B + C - 2D.$$

Thus the area of the triangle of which the sides are a, b, c and the angles A, B, C ſhall be the $\frac{\Sigma}{8D}$ part of the whole ſurface of the ſphere; and if this ſurface be conſidered as equal to 8D, we may take Σ for the area of the triangle.

9. If we ſuppoſe the sides b and c, which contain the angle A, to be produced till each of them be a quadrant, the angles B, C will then become right angles, and the side a will become equal to the oppoſite angle A; then the area of the iſoſceles right angled triangle becomes A; therefore, if from it we ſubtract the former triangle, of which the sides about the angle A are b and c, we ſhall have a ſpherical quadrilateral, the baſe of which is A, and the sides perpendicular to that baſe $D - b$ and $D - c$, and the area of the quadrilateral ſhall be expreſſed simply by $B + C - 2D$.

But by the known analogies of Napier, we have in every ſpherical triangle this equation, which will be demonſtrated farther on,

ſpherical triangle, the angles of the latter ſhall be equal to the correſponding angles of the rectilineal triangle, when each is augmented by the quantity $\frac{\theta}{3r^2}$, (θ being the area of the rectilineal triangle) reduced to the meaſure of an angle, which is done by taking for unity the angle correſponding to an arch equal to the radius.

28. If we add together the three equations $A = A' + \frac{\theta}{3r^2}$, $B = B' + \frac{\theta}{3r^2}$, $C = C' + \frac{\theta}{3r^2}$, we have $A + B + C = A' + B' + C' + \frac{\theta}{r^2}$, but we know that $A' + B' + C' = 2D$ where D denotes a right angle; therefore,

$$\frac{\theta}{r^2} = A + B + C - 2D.$$

Hence we may conclude, that by ſubtracting from each angle of a ſpherical triangle one-third of the exceſs of the ſum of the three angles above two right angles, we ſhall have the three angles of a rectilineal triangle, the ſides of which are equal in length to thoſe of the ſpherical triangle. Thus we may treat the latter as a rectilineal triangle, and obtain reſults which will be accurate to quantities nearly of the order $\frac{1}{r^4}$.

This beautiful theorem is due to Legendre, who gave it at firſt without demonſtration in the Memoirs of the Academy of Sciences for 1787; he has ſince demonſtrated it, in a manner differing but little from the preceding, in a Memoir upon the method of determining the length of a quadrant of the meridian. As it may be of great uſe in the calculation of ſpherical triangles which differ but little from rectilineal triangles, we have judged that the reader would be pleaſed to find it here.

If we ſubſtitute, in place of tan. $\frac{B+C}{2}$ its value found above (preceding No.), we have

$$\text{cot.}\,\frac{\Sigma}{2} = -\frac{\text{tan.}\,\frac{A}{2}\,\text{coſ.}\,\frac{b+c}{2} + \text{cot.}\,\frac{A}{2}\,\text{coſ.}\,\frac{b-c}{2}}{\text{coſ.}\,\frac{b+c}{2} - \text{coſ.}\,\frac{b-c}{2}},$$

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ſimple applications of algebra to geometrical conſtructions. It was afterwards thought ſufficient to eſtabliſh ſome fundamental propoſitions by the aid of geometry, and all the formulas of ſpherical trigonometry have been deduced from equations derived from thoſe propoſitions. The moſt elegant production of this kind is a Memoir of Euler's, entitled *Trigonometria ſpherica univerſa ex primis principiis derivata*, and printed in the Petersburgh Acts for the year 1779, in which there is found a complete ſyſtem of trigonometrical formulas, founded ſolely upon three equations. But may not the ſyſtem be yet more ſimplified by reducing it to one fundamental equation?

Such a reduction will complete the analytical theory of ſpherical triangles; for, in Analyſis, perfection conſiſts in employing the ſmalleſt poſſible number of principles, and in deducing from theſe all the truths they contain by the powers of that method alone: on the contrary, in the ſynthetic method of lines, it conſiſts in demonſtrating ſeparately every propoſition in the moſt ſimple manner, by means of other propoſitions previouſly demonſtrated.

The late de Gua had conceived the idea of making all ſpherical trigonometry to depend upon a ſingle general property of ſpherical triangles; but the Memoir he has given upon the ſubject, in the volume of the Academy of Sciences for 1783, contains calculations ſo complicated, that they ſeem better adapted to ſhew the inconvenience of his method than to cauſe it to be adopted.

I here propoſe to myſelf the ſame object, and I proceed to give a conciſe view of all the formulas of ſpherical trigonometry, deducing them by ſimple transformations from a ſingle equation, given by the nature of ſpherical triangles.

Let us aſſume, as De Gua has done, the equation (No. 3.)

$$\text{coſ.}\, a = \text{coſ.}\, b\ \text{coſ.}\, c + \text{ſin.}\, b\ \text{ſin.}\, c\ \text{coſ.}\, A$$

in which a, b, c are the three ſides or arches of the triangle, and A the angle oppoſite to the ſide a.

This equation is eaſily demonſtrated by the ſimple conſideration of two rectilineal triangles formed, the one by the two tangents to the arches b, c, and by the ſtraight line which joins the extremities of theſe tangents, and the other by the ſame line and the two ſecants of the ſame arches; for it is obvious, that the two tangents form between them the angle A which is contained by the arches b and c, and that the two ſecants form between them the angle a which is the ſide of the ſpherical triangle oppoſite to the angle A. Thus, calling h the ſide common to theſe two triangles, we have immediately, by a known theorem concerning rectilineal triangles, this equation,

$$h^2 = \text{tan.}^2 b + \text{tan.}^2 c - 2\text{tan.}\, b\ \text{tan.}\, c\ \text{coſ.}\, A$$

for the former triangle, and this other equation,

$$h^2 = \text{ſec.}^2 b + \text{ſec.}^2 c - 2\text{ſec.}\, b\ \text{ſec.}\, c\ \text{coſ.}\, a$$

for the latter triangle.

Hence we find

$$\text{tan.}^2 b + \text{tan.}^2 c - 2\,\text{tan.}\,b\,\text{tan.}\,c\,\text{coſ.}\,A$$
$$= \text{ſec.}^2 b + \text{ſec.}^2 c - 2\,\text{ſec.}\,b\,\text{ſec.}\,c\,\text{coſ.}\,a.$$

But we have evidently

$$\text{ſec.}^2 b - \text{tan.}^2 b = 1, \text{ alſo } \text{ſec.}^2 c - \text{tan.}^2 c = 1,$$

therefore, the equation becomes

$$\text{ſec.}\,b\,\text{ſec.}\,c\,\text{coſ.}\,a = 1 + \text{tan.}\,b\,\text{tan.}\,c\,\text{coſ.}\,A,$$

hence, ſubſtituting for ſec. b, ſec. c, tan. b, tan. c, their values $\frac{1}{\text{coſ.}\,b}$, $\frac{1}{\text{coſ.}\,c}$, $\frac{\text{sin.}\,b}{\text{coſ.}\,b}$, $\frac{\text{sin.}\,c}{\text{coſ.}\,c}$, we have the fundamental equation,

$$\text{coſ.}\,a = \text{coſ.}\,b\,\text{coſ.}\,c + \text{sin.}\,b\,\text{sin.}\,c\,\text{coſ.}\,A \ldots\ldots\ldots\ldots (A)$$

As, by hypotheſis, the only condition ſuppoſed to have place among the four quantities a, b, c, A is that a, b, c are the three ſides of a triangle, and A the angle oppoſite to the ſide a; it follows, that by putting B and C for the angles oppoſite to the ſides b and c, we ſhall have ſimilar equations relatively to thoſe angles, by ſimply changing A into B, or into C, provided that at the ſame time we change a into b, or into c.

15. Now if we deduce the value of coſ. A from the preceding equations, and thence that of sin. A we ſhall have, as has been already found in No. 3,

$$\frac{\text{sin.}\,a}{\text{sin.}\,A} = \frac{\text{sin.}\,a\,\text{sin.}\,b\,\text{sin.}\,c}{f}$$

where the quantity f is a function of a, b, c into which theſe three quantities enter ſimilarly, ſo that it remains the ſame, whatever permutations be made among them. Therefore, in changing a into b, and b into a, the ſecond member of the equation will be the ſame as before; hence we have this other equation,

$$\frac{\text{sin.}\,a}{\text{sin.}\,A} = \frac{\text{sin.}\,b}{\text{sin.}\,B} \ldots\ldots\ldots\ldots (B)$$

This is what is called the common analogy of the sines, and it is obvious, that by changing a and c into A and C, we have in like manner $\frac{\text{sin.}\,a}{\text{sin.}\,A} = \frac{\text{sin.}\,c}{\text{sin.}\,C}$.

16. Let us reſume the equation (A) of No. 14, $\text{coſ.}\,a = \text{coſ.}\,b\,\text{coſ.}\,c + \text{sin.}\,b\,\text{sin.}\,c\,\text{coſ.}\,A$; by changing a into c, and A into C, we have in like manner $\text{coſ.}\,c = \text{coſ.}\,a\,\text{coſ.}\,b + \text{sin.}\,a\,\text{sin.}\,b\,\text{coſ.}\,C$; ſubſtituting this value of coſ. c in the firſt equation it becomes $\text{coſ.}\,a = \text{coſ.}\,a\,\text{coſ.}^2 b + \text{sin.}\,a\,\text{sin.}\,b\,\text{coſ.}\,b\,\text{coſ.}\,C$

therefore $\text{coſ. } A = \frac{b^2 + c^2 - a^2}{2bc}$, and hence

$$\text{ſin. } A = \frac{\sqrt{[4b^2c^2 - (b^2 + c^2 - a^2)^2]}}{2bc}.$$

Therefore, if in order to ſimplify, we put

$$d = \sqrt{[4b^2c^2 - (b^2 + c^2 - a^2)^2]}$$

we have $\text{ſin. } A = \frac{d}{2bc}$, and hence $\frac{a}{\text{ſin. } A} = \frac{2abc}{d}$ which fraction expreſſes the ratio ſought.

If r be put for the radius of the circle which circumſcribes the triangle, and s for the ſurface, or area of the ſame triangle, we have $\frac{a}{\text{ſin. } A} = \frac{2abc}{d} = 2r = \frac{abc}{2s}$; therefore $r = \frac{abc}{d}$, and and $s = \frac{abc}{4r} = \frac{d}{4}$.

2. By developing the ſquare of $b^2 + c^2 - a^2$, and reducing, we have

$$d = \sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)}$$

a formula in which the three ſides a, b, c are equally concerned, as they ought to be.

But this formula may be rendered more ſimple and more commodious for logarithmic calculation by reſolving it into factors. For

$$4b^2c^2 - (b^2 + c^2 - a^2)^2 = (2bc + b^2 + c^2 - a^2)\,(2bc - b^2 - c^2 + a^2),$$

and again reſolving each of theſe factors into two, we have

$$d = \sqrt{[(a + b + c)\,(a - b + c)\,(a + b - c)\,(-a + b + c)]}.$$

Theſe formulas are already known, and I only give them here as an introduction to inveſtigations which are to follow.

3. Since in ſpherical triangles the ſines of the ſides are proportional to the ſines of the angles oppoſite to them, we may be deſirous to inveſtigate the conſtant ratio, and to diſcover if it depends, as in rectilineal triangles, upon the radius of the circumſcribed circle, or the area of the triangle.

Let us denote as before the three ſides of a ſpherical triangle by a, b, c, and the three angles oppoſite to thoſe ſides by A, B, C.

Then by a known theorem

$$\text{coſ. } a = \text{coſ. } b \text{ coſ. } c + \text{ſin. } b \text{ ſin. } c \text{ coſ. } A;$$

therefore $\text{coſ. } A = \frac{\text{coſ. } a - \text{coſ. } b \text{ coſ. } c}{\text{ſin. } b \text{ ſin. } c}$; and hence

$$\text{ſin. } A = \frac{\sqrt{[\text{ſin.}^2 b \text{ ſin.}^2 c - (\text{coſ. } a - \text{coſ. } b \text{ coſ. } c)^2]}}{\text{ſin. } b \text{ ſin. } c}$$

Let us, to abreviate, put

$$f = \sqrt{[\text{ſin.}^2 b\ \text{ſin.}^2 c - (\text{coſ.}\ a - \text{coſ.}\ b\ \text{coſ.}\ c)^2]}$$

thus we have ſin. A $= \dfrac{f}{\text{ſin.}\ b\ \text{ſin.}\ c}$, and therefore

$$\frac{\text{ſin.}\ a}{\text{ſin.}\ A} = \frac{\text{ſin.}\ a\ \text{ſin.}\ b\ \text{ſin.}\ c}{f}$$

an expreſſion for the ratio ſought analogous to that which has been already found for rectilineal triangles (No. 1).

As the quantity f is expreſſed by a ſquare root, and may, therefore, have either the ſign plus, or minus prefixed to it, we remark, that relatively to ſpherical triangles, it ought to be always poſitive; for the ſides and the angles of every triangle being always leſs than two right angles, the ſines are neceſſarily always poſitive.

4. The radical quantity f is alſo ſuſceptible of reductions ſimilar to thoſe of No. 2. For by ſubſtituting inſtead of ſin.² b and ſin.² c their values $1 - \text{coſ.}^2 b$ and $1 - \text{coſ.}^2 c$; and afterwards reducing, we have

$$f = \sqrt{[1 - \text{coſ.}^2 a - \text{coſ.}^2 b - \text{coſ.}^2 c + 2\text{coſ.}\ a\ \text{coſ.}\ b\ \text{coſ.}\ c]}$$

where it appears that the three ſides a, b, c are alike concerned.

In like manner the quantity under the radical ſign may be reſolved into factors, for we have preſently

$$\begin{aligned}&\text{ſin.}^2 b\ \text{ſin.}^2 c - (\text{coſ.}\ a - \text{coſ.}\ b\ \text{coſ.}\ c)^2\\ &= (\text{ſin.}\ b\ \text{ſin.}\ c + \text{coſ.}\ a - \text{coſ.}\ b\ \text{coſ.}\ c)\\ &\quad\times (\text{ſin.}\ b\ \text{ſin.}\ c - \text{coſ}\ a + \text{coſ.}\ b\ \text{coſ.}\ c)\\ &= [\text{coſ.}\ a - \text{coſ.}\ (b + c)]\ [\text{coſ.}\ (b - c) - \text{coſ.}\ a]\end{aligned}$$

but we have, in general $\text{coſ.}\ a - \text{coſ.}\ h = 2\ \text{ſin.}\ \dfrac{a+h}{2} \times \text{ſin.}\ \dfrac{h-a}{2}$

therefore, reſolving thus the two factors, we have

$$f = 2\sqrt{[\text{ſin.}\frac{a+b+c}{2} \times \text{ſin.}\frac{a-b+c}{2} \times \text{ſin.}\frac{a+b-c}{2} \times \text{ſin.}\frac{-a+b+c}{2}]}$$

an expreſſion very convenient for logarithmic calculation.

5. Let us now ſeek the radius of the circle circumſcribing the ſpherical triangle. It is evident that this circle muſt be a ſmall circle of the ſphere, and that it will alſo circumſcribe the rectilineal triangle formed by the chords of the arches a, b, c; but theſe chords being expreſſed by $2\ \text{ſin.}\ \frac{a}{2}$, $2\text{ſin.}\ \frac{b}{2}$, $2\text{ſin.}\ \frac{2}{c}$ it is only neceſſary to ſubſtitute theſe quantitities inſtead of a, b,

c in the expreſſion of the radius r (No. 2), that is to ſay, in $\frac{abc}{d}$.

Let us put R for the radius of the circumſcribed circle, and h for the value which d acquires by ſubſtituting for a, b, c as above; thus we have

$$R = \frac{8\,\text{ſin.}\,\frac{a}{2} \times \text{ſin.}\,\frac{b}{2} \times \text{ſin.}\,\frac{c}{2}}{h}.$$

But we have $h^2 = 32\,(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{b}{2})^2 + 32\,(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{c}{2})^2 + 32\,(\text{ſin.}\,\frac{b}{2}\,\text{ſin.}\,\frac{c}{2})^2 - 16\,(\text{ſin.}\,\frac{a}{2})^4 - 16(\text{sin.}\,\frac{b}{2})^4 - 16(\text{sin.}\,\frac{c}{2})^4$;

and ſubſtituting for $2(\text{sin.}\,\frac{a}{2})^2$, $2(\text{sin.}\,\frac{b}{2})^2$, $2(\text{sin.}\,\frac{c}{2})^2$ their values $1 - \text{coſ.}\,a$, $1 - \text{coſ.}\,b$, $1 - \text{coſ.}\,c$ we have after proper reductions

$$\begin{aligned} h^2 = {} & 12 - 8\,\text{coſ.}\,a - 8\,\text{coſ.}\,b - 8\,\text{coſ.}\,c \\ & + 8\,\text{coſ.}\,a\,\text{coſ.}\,b + 8\,\text{coſ.}\,a\,\text{coſ.}\,c + 8\,\text{coſ.}\,b\,\text{coſ.}\,c \\ & - 4\,\text{coſ.}^2\,a - 4\,\text{coſ.}^2\,b - 4\,\text{coſ.}^2\,c \end{aligned}$$

which expreſſion may be reduced to the following

$$4f^2 + 8(1 - \text{coſ.}\,a)\,(1 - \text{coſ.}\,b)\,(1 - \text{coſ.}\,c)$$

that is to ſay, to

$$4f^2 + 64\,(\text{ſin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2})^2$$

by ſubſtituting the value of f of No. 4. Thus we have

$$R = \frac{4\,\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2}}{\sqrt{[f^2 + 16(\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2})^2]}}.$$

6. Now by conſidering the radius of the ſphere which paſſes through the centre of the ſmall circle circumſcribing the triangle, it is evident that this radius is perpendicular to the plane of the circle, and that it meets the ſurface of the ſphere at a point which is the pole of the ſame circle.

Therefore, putting φ for the arch which meaſures the diſtance of the circle from its pole, we have evidently $R = \text{sin.}\,\varphi$, therefore

sin.

hence we have this transformation,

$$\frac{B}{A} = \frac{m}{a} \pm \frac{1}{Aa},$$

where the upper or lower ſign is to be taken according as the given fraction is greater or leſs than the new fraction $\frac{m}{a}$.

15. But it is neceſſary to be firſt aſſured that two numbers $a < A$, $m < B$ may be found, ſuch that $Ba - Am = \pm 1$. Let us put this equation under the form $a = \frac{Am \pm 1}{B}$; then, that this may be poſſible, it is evident that we muſt find a number m leſs than B, which ſhall render the number $mA \pm 1$ diviſible by B. Now, if in $mA \pm 1$ there be ſubſtituted ſucceſſively for m, all the numbers 0, 1, 2, &c. to $B - 1$, and each reſult be divided by B, the remainders will be all leſs than B, and different from each other; for if there can be two equal remainders, let m and m' be the two numbers which give the ſame remainder, then the difference $(m - m')$ A will be diviſible by B, but A and B are prime to each other, and $m - m'$ is a number leſs than B, ſeeing that m and m' are leſs than B, therefore this difference cannot be diviſible by B; hence it follows, that the two remainders cannot be equal, therefore 0 muſt neceſſarily be found among the remainders; and, conſequently, there may be always a number leſs than B, which ſubſtituted for m in $mA \pm 1$ ſhall render this laſt number diviſible by B. The ſame number may, therefore, be taken for m, and the quotient of the diviſion of $Am \pm 1$ by B will be the correſponding value of a, which will, conſequently, be leſs than A.

It appears alſo by this demonſtration that there can only be one value of m and one of a, leſs than B and A, which will ſatisfy the equation $Ba - Am = \pm 1$. For let a and a' be two values of a, and m, m' two values of m, we have, therefore, $Ba - Am = \pm 1$, $Ba' - Am' = \pm 1$, and, by ſubtracting the one equation from the other, $B(a - a') = A \cdot m - m')$; but this equation cannot take place in whole numbers, for A and B are prime to each other, and $a - a'$ and $m - m'$ are numbers leſs than A and B. Therefore, two numbers m and a may always be found ſo as to ſatisfy the equation, by repeated trials of numbers leſs than B or A for m or a; but we ſhall afterwards give direct methods for finding them.

16. Again, when two values of m and a have been found ſo as to ſatisfy the equation $Ba - Am = \pm 1$, if we take $a' = A - a$, $m' = B - m$, we ſhall have $Ba' - Am' = \mp 1$, thus it appears that A is ſufficient, if ſuch values of m and a be found as to ſatisfy the propoſed equation, when the ambiguous ſign is taken either as

poſitive

$$\frac{2\sin.\frac{a}{2}\ \sin.\frac{b}{2}\ \sin.\frac{c}{2}}{3\tan.\ \varphi}$$

and again, by ſubſtituting for tan. φ its value already found (No. 6), we have $\frac{\mathcal{L}}{6}$ for the ſolidity of the pyramid.

8. It remains for us to conſider the area of the ſpherical triangle itſelf formed by the arcs *a*, *b*, *c*. There is a beautiful and known theorem, according to which, the area of a ſpherical triangle is to the whole ſurface of the ſphere, as the exceſs of the three angles of the triangle above two right angles to eight right angles. This theorem is commonly attributed to Albert Girard, who gave it in ſubſtance, in a work entitled *Invention Nouvelle en Algebre*, printed at Amſterdam in 1629; but as the proof which he gives is not rigorous, and cannot even be conſidered as an induction, the theorem ought rather to be attributed to Cavallerius, who gave it in the *Directericum generale uranometricum*, printed at Bologna in 1632, with the elegant demonſtration firſt publiſhed by Wallis, and since inſerted in moſt works upon trigonometry.

Let Σ denote the exceſs of the three angles of the triangle above two right angles, then, retaining the notation hitherto employed, and putting D for a right angle,

$$\Sigma = A + B + C - 2D.$$

Thus the area of the triangle of which the sides are *a*, *b*, *c* and the angles A, B, C ſhall be the $\frac{\Sigma}{8D}$ part of the whole ſurface of the ſphere; and if this ſurface be conſidered as equal to 8D, we may take Σ for the area of the triangle.

9. If we ſuppoſe the sides *b* and *c*, which contain the angle A, to be produced till each of them be a quadrant, the angles B, C will then become right angles, and the side *a* will become equal to the oppoſite angle A; then the area of the iſoſceles right angled triangle becomes A; therefore, if from it we ſubtract the former triangle, of which the sides about the angle A are *b* and *c*, we ſhall have a ſpherical quadrilateral, the baſe of which is A, and the sides perpendicular to that baſe D — *b* and D — *c*, and the area of the quadrilateral ſhall be expreſſed simply by B + C — 2D.

But by the known analogies of Napier, we have in every ſpherical triangle this equation, which will be demonſtrated farther on,

tan.

$$\tan.\frac{B+C}{2}=\frac{\text{coſ.}\,\frac{b-c}{2}}{\text{coſ.}\,\frac{b+c}{2}}\cot.\frac{A}{2}.$$

Therefore, if σ denote the ſurface of the quadrilateral we are conſidering, we ſhall have

$$\tan.\frac{\sigma}{2}=\cot.\frac{B+C}{2}=\frac{1}{\tan.\frac{B+C}{2}};\text{ therefore,}$$

$$\tan.\frac{\sigma}{2}=\frac{\text{coſ.}\,\frac{b+c}{2}}{\text{coſ.}\,\frac{b-c}{2}}\tan.\frac{A}{2};$$

and if we put β and γ for the two ſides of the quadrilateral perpendicular to the baſe A, ſo that $\beta=D-b$ and $\gamma=D-c$ we have in order to determine the area σ this formula

$$\tan.\frac{\sigma}{2}=\frac{\sin.\frac{\beta+\gamma}{2}}{\text{coſ.}\,\frac{\beta-\gamma}{2}}\tan.\frac{A}{2}.$$

This formula correſponds to the known formula $\sigma=\frac{\beta+\gamma}{2}A$ for a rectilineal quadrilateral of which A is the baſe, β and γ the two vertical ſides, and σ the area; and as the latter is of the greateſt utility in the menſuration of plane ſurfaces bounded by ſtraight lines, ſo the former is equally uſeful in the menſuration of ſpherical ſurfaces bounded by arches of great circles. Thus it may be employed with much advantage to determine the extent of a country, when the latitudes and longitudes of ſeveral points in its circumference are known; for by connecting thoſe points by arches of great circles, we have a ſpherical polygon, the area of which may be eaſily found, by reſolving it into quadrilaterals formed by circles of latitudes and arches of the equator intercepted between thoſe circles.

10. But if it was required to expreſs the area Σ by the three ſides a, b, c of the ſpherical triangle, it is only neceſſary to conſider, that ſince $\Sigma=A+B+C-2D$ we have

$$\cot.\frac{\Sigma}{2}=-\tan.\frac{A+B+C}{2}$$

$$=-\frac{\tan.\frac{A}{2}+\tan.\frac{B+C}{2}}{1-\tan.\frac{A}{2}\tan.\frac{B+C}{2}}.$$

$$\frac{2\sin.\frac{a}{2}\sin.\frac{b}{2}\sin.\frac{c}{2}}{3\tan.\varphi}$$

and again, by ſubſtituting for tan. φ its value already found (No. 6), we have $\frac{f}{6}$ for the ſolidity of the pyramid.

8. It remains for us to conſider the area of the ſpherical triangle itſelf formed by the arcs *a*, *b*, *c*. There is a beautiful and known theorem, according to which, the area of a ſpherical triangle is to the whole ſurface of the ſphere, as the exceſs of the three angles of the triangle above two right angles to eight right angles. This theorem is commonly attributed to Albert Girard, who gave it in ſubſtance, in a work entitled *Invention Nouvelle en Algebre*, printed at Amſterdam in 1629; but as the proof which he gives is not rigorous, and cannot even be conſidered as an induction, the theorem ought rather to be attributed to Cavallerius, who gave it in the *Directericum generale uranometricum*, printed at Bologna in 1632, with the elegant demonſtration firſt publiſhed by Wallis, and ſince inſerted in moſt works upon trigonometry.

Let Σ denote the exceſs of the three angles of the triangle above two right angles, then, retaining the notation hitherto employed, and putting D for a right angle,

$$\Sigma = A + B + C - 2D.$$

Thus the area of the triangle of which the ſides are *a*, *b*, *c* and the angles A, B, C ſhall be the $\frac{\Sigma}{8D}$ part of the whole ſurface of the ſphere; and if this ſurface be conſidered as equal to 8D, we may take Σ for the area of the triangle.

9. If we ſuppoſe the ſides *b* and *c*, which contain the angle A, to be produced till each of them be a quadrant, the angles B, C will then become right angles, and the ſide *a* will become equal to the oppoſite angle A; then the area of the iſoſceles right angled triangle becomes A; therefore, if from it we ſubtract the former triangle, of which the ſides about the angle A are *b* and *c*, we ſhall have a ſpherical quadrilateral, the baſe of which is A, and the ſides perpendicular to that baſe D — *b* and D — *c*, and the area of the quadrilateral ſhall be expreſſed ſimply by B + C — 2D.

But by the known analogies of Napier, we have in every ſpherical triangle this equation, which will be demonſtrated farther on,

$A - b$ are equimultiples of m and a; we have therefore, taking for λ a whole indeterminate number, $B - n = \lambda m$, $A - b = \lambda a$; therefore, $n = B - \lambda m$, $b = A - \lambda a$.

But as n ought to be $< m$, and $b < a$, it is evident that we can only take for n and b the remainders of the diviſions of B by m, and of A by a, and λ will then be the common quotient of theſe diviſions. Thus, knowing the two firſt fractions $\frac{B}{A}$, $\frac{m}{a}$, we may find, in this manner, the third $\frac{n}{b}$. In like manner the equations $mb - an = \mp 1$, and $nc - bp = \pm 1$ being added together give $(m - p)\, b - (a - c)\, n = 0$, that is $\frac{m - p}{a - c} = \frac{n}{b}$; hence we find as before $m - p = \mu n$, $a - c = \mu b$, μ being any whole number. We have thus $p = m - \mu n$ $c = a - \mu b$, and as p ought to be $< n$, and $c < b$, it follows that p and c can only be the remainders of the diviſions of m by n, and of a by b, and that μ is their common quotient.

In the ſame way, from the two equations $nc - pb = \pm 1$, $pd - cq = \mp 1$, we derive theſe two formulas $q = n - \nu p$, $d = b - \nu c$, ν being a whole indeterminate number. And as q and d ought to be reſpectively leſs than p and e, we thence conclude that they can only be the remainders of the diviſions of n by p, and of b by e, and that ν is their common quotient, and ſo on.

21. We have found the formulas

$$n = B - \lambda m,\quad p = m - \mu n,\quad q = n - \nu p,\ \&c.$$
$$b = A - \lambda a,\quad c = a - \mu b,\quad d = b - \nu c,\ \&c.$$

If theſe values be ſubſtituted in the expreſſions of the numbers N, P, Q, &c. of No. 19, it will be found, becauſe of $Ba - Am = \pm 1$, and by taking away the ambiguous ſign which affects all the terms, that

$$N = \lambda,\quad P = 1 + \mu N,\quad Q = N + \nu P,\quad R = P + \pi Q,$$
$$S = Q + \rho R,\ \&c.$$

theſe formulas ſhew that the numbers N, P, Q, &c. neceſſarily go on increaſing, while the numbers a, b, c, d, &c. go on decreaſing. Theſe formulas may alſo ſerve to determine directly the value of the numbers when the coefficients λ, μ, ν, &c. are known, and the ſame numbers N, P, Q, &c. ſerve to expreſs in a ſimple manner the differences of the fractions $\frac{m}{a}$, $\frac{n}{b}$, &c. and of the fraction $\frac{B}{A}$, (No. 19).

22. Now as the numbers m, n, p, &c. alſo the numbers a, b, c, &c. muſt go on decreaſing, it is evident, that by con-

If we ſubſtitute, in place of tan. $\frac{B+C}{2}$ its value found above (preceding No.), we have

$$\text{cot.}\ \frac{\Sigma}{2} = -\frac{\text{tan.}\ \frac{A}{2}\,\text{coſ.}\ \frac{b+c}{2} + \text{cot.}\ \frac{A}{2}\,\text{coſ.}\ \frac{b-c}{2}}{\text{coſ.}\ \frac{b+c}{2} - \text{coſ.}\ \frac{b-c}{2}},$$

a formula, which is eaſily transformed to this other

$$\text{cot.}\ \frac{\Sigma}{2} = \frac{\text{coſ.}\ \frac{b}{2}\,\text{coſ.}\ \frac{c}{2} + \text{ſin.}\ \frac{b}{2}\,\text{ſin.}\ \frac{c}{2}\,\text{coſ. A}}{\text{ſin.}\ \frac{b}{2}\,\text{ſin.}\ \frac{c}{2}\,\text{ſin. A}}.$$

If now we ſubſtitute in this formula the values ſin. A, and coſ. A of No. 3, we ſhall have, by dividing the numerator and denominator by ſin. $\frac{b}{2}$ ſin. $\frac{c}{2}$

$$\text{cot.}\ \frac{\Sigma}{2} = \frac{4(\text{coſ.}\ \frac{b}{2}\,\text{coſ.}\frac{c}{2})^2 + \text{coſ.}\,a - \text{coſ.}\ b\ \text{coſ.}\ c}{f};$$

But $2(\text{coſ.}\ \frac{b}{2})^2 = 1 + \text{coſ.}\ b$, and $2(\text{coſ.}\ \frac{c}{2})^2 = 1 + \text{coſ.}\,c$, therefore, making the neceſſary ſubſtitutions and inverting the fraction,

$$\text{tan.}\ \frac{\Sigma}{2} = \frac{f}{1 + \text{coſ.}\ a + \text{coſ.}\ b + \text{coſ.}\ c}$$

a formula of the moſt ſimple form to determine the area of a ſpherical triangle by means of its three ſides a, b, c.

11. We have ſeen (No. 7) that $\frac{f}{6}$ is the ſolidity of the triangular pyramid formed by the three radii of the ſphere which terminate in the angles of a ſpherical triangle. Let us now conſider a triangular pyramid formed by the ſame radii produced at pleaſure in ſuch a manner that they become p, q, r; and that a, b, c are the arches, or angles contained by thoſe ſtraight lines. To have the ſolidity of this pyramid, it may be conſidered as lying on one of its faces; for inſtance, that which has for its ſides the lines p, q, and to have a perpendicular P drawn from the other ſide r upon the plane of that face. It is immediately obvious, that if a is the angle contained by p and q, the area of the face which we have conſidered as the baſe of the

pyramid will be $\frac{pq\,\text{ſin.}\,a}{2}$, therefore, the ſolidity of the pyramid will be $\frac{Ppq\,\text{ſin.}\,a}{6}$. But if we call θ the angle which the line r makes with the plane paſſing through p and q, it is evident that we have $P = r\,\text{ſin.}\,\theta$, therefore, the ſolidity required will be $\frac{pqr\,\text{ſin.}\,a\,\text{ſin.}\,\theta}{6}$.

The angle θ is no other than the arch let fall perpendicularly from the angle A of the ſpherical triangle upon the oppoſite ſide a; we may, therefore, expreſs ſin. θ by the ſines or coſines of a, b, c the ſides of the triangle; but for our purpoſe it is ſufficient to conſider that this value, as well as that of ſin. a, being independent of the lines p, q, r, if we make $p = 1, q = 1, r = 1$ we ſhall have the caſe of the particular pyramid we propoſe to conſider; and of which the ſolidity is $\frac{f}{6}$. Hence it follows, that we have ſin. a ſin. $\theta = f$, and therefore $\frac{pqrf}{6}$ for the ſolidity of the triangular pyramid in which the three ſides, or edges that form any one of the ſolid angles are p, q, r, and the angles contained by thoſe three ſides are A, B, C.

This expreſſion, for the ſolidity of any triangular pyramid, by means of its three ſides and contained angles, is evidently very ſimple and commodious for calculation, more eſpecially if we employ the value of f, as expreſſed by factors, in No. 4; and it may be very uſeful to determine the ſolid contents of all bodies bounded by planes, ſeeing that they may always be reſolved into triangular pyramids, as a polygon is reſolved into triangles.

12. We may add, that ſince we have found ſin. a ſin. $\theta = f$, we ſhall have ſin. $\theta = \frac{f}{\text{ſin.}\,a}$. We may thus determine by this formula the perpendicular θ in a ſpherical triangle, of which a is the baſe, and b, c the two ſides.

13. I have reſolved the preceding problems, only to ſhew the origin and uſe of ſome remarkable formulas, and more eſpecially of that function which I have denoted by f, and which deſerves in a particular manner the attention of analyſts, on account of its different applications. I now proceed to general conſiderations relating to ſpherical trigonometry, viewed analytically.

Analytical reſolutions of ſpherical triangles were at firſt only

ſimple applications of algebra to geometrical conſtructions. It was afterwards thought ſufficient to eſtabliſh ſome fundamental propoſitions by the aid of geometry, and all the formulas of ſpherical trigonometry have been deduced from equations derived from thoſe propoſitions. The moſt elegant production of this kind is a Memoir of Euler's, entitled *Trigonometria ſpherica univerſa ex primis principiis derivata*, and printed in the Petersburgh Acts for the year 1779, in which there is found a complete ſyſtem of trigonometrical formulas, founded ſolely upon three equations. But may not the ſyſtem be yet more ſimplified by reducing it to one fundamental equation?

Such a reduction will complete the analytical theory of ſpherical triangles; for, in Analyſis, perfection conſiſts in employing the ſmalleſt poſſible number of principles, and in deducing from theſe all the truths they contain by the powers of that method alone: on the contrary, in the ſynthetic method of lines, it conſiſts in demonſtrating ſeparately every propoſition in the moſt ſimple manner, by means of other propoſitions previouſly demonſtrated.

The late de Gua had conceived the idea of making all ſpherical trigonometry to depend upon a ſingle general property of ſpherical triangles; but the Memoir he has given upon the ſubject, in the volume of the Academy of Sciences for 1783, contains calculations ſo complicated, that they ſeem better adapted to ſhew the inconvenience of his method than to cauſe it to be adopted.

I here propoſe to myſelf the ſame object, and I proceed to give a conciſe view of all the formulas of ſpherical trigonometry, deducing them by ſimple transformations from a ſingle equation, given by the nature of ſpherical triangles.

Let us aſſume, as De Gua has done, the equation (No. 3.)

$$\text{coſ. } a = \text{coſ. } b \text{ coſ. } c + \text{ſin. } b \text{ ſin. } c \text{ coſ. A}$$

in which a, b, c are the three ſides or arches of the triangle, and A the angle oppoſite to the ſide a.

This equation is eaſily demonſtrated by the ſimple conſideration of two rectilineal triangles formed, the one by the two tangents to the arches b, c, and by the ſtraight line which joins the extremities of theſe tangents, and the other by the ſame line and the two ſecants of the ſame arches; for it is obvious, that the two tangents form between them the angle A which is contained by the arches b and c, and that the two ſecants form between them the angle a which is the ſide of the ſpherical triangle oppoſite to the angle A. Thus, calling h the ſide common to theſe two triangles, we have immediately, by a known theorem concerning rectilineal triangles, this equation,

$$h^2 = \text{tan.}^2 b + \text{tan.}^2 c - 2\text{tan. } b \text{ tan. } c \text{ coſ. A}$$

for the former triangle, and this other equation,

$$h^2 = \text{ſec.}^2 b + \text{ſec.}^2 c - 2\text{ſec. } b \text{ ſec. } c \text{ coſ. } a$$

for the latter triangle.

Hence we find

$$\text{tan.}^2\, b + \text{tan.}^2\, c - 2\,\text{tan.}\, b\, \text{tan.}\, c\, \text{coſ.}\, A$$
$$= \text{ſec.}^2\, b + \text{ſec.}^2\, c - 2\,\text{ſec.}\, b\, \text{ſec.}\, c\, \text{coſ.}\, a.$$

But we have evidently

$$\text{ſec.}^2\, b - \text{tan.}^2\, b = 1,\ \text{alſo}\ \text{ſec.}^2\, c - \text{tan.}^2\, c = 1,$$

therefore, the equation becomes

$$\text{ſec.}\, b\, \text{ſec.}\, c\, \text{coſ.}\, a = 1 + \text{tan.}\, b\, \text{tan.}\, c\, \text{coſ.}\, A,$$

hence, ſubſtituting for ſec. b, ſec. c, tan. b, tan. c, their values $\frac{1}{\text{coſ.}\, b}$, $\frac{1}{\text{coſ.}\, c}$, $\frac{\text{ſin.}\, b}{\text{coſ.}\, b}$, $\frac{\text{ſin.}\, c}{\text{coſ.}\, c}$, we have the fundamental equation,

$$\text{coſ.}\, a = \text{coſ.}\, b\, \text{coſ.}\, c + \text{ſin.}\, b\, \text{ſin.}\, c\, \text{coſ.}\, A \ldots\ldots\ldots\ldots (A)$$

As, by hypotheſis, the only condition ſuppoſed to have place among the four quantities a, b, c, A is that a, b, c are the three ſides of a triangle, and A the angle oppoſite to the ſide a; it follows, that by putting B and C for the angles oppoſite to the ſides b and c, we ſhall have ſimilar equations relatively to thoſe angles, by ſimply changing A into B, or into C, provided that at the ſame time we change a into b, or into c.

15. Now if we deduce the value of coſ. A from the preceding equations, and thence that of ſin. A we ſhall have, as has been already found in No. 3,

$$\frac{\text{ſin.}\, a}{\text{ſin.}\, A} = \frac{\text{ſin.}\, a\, \text{ſin.}\, b\, \text{ſin.}\, c}{f}$$

where the quantity f is a function of a, b, c into which theſe three quantities enter ſimilarly, ſo that it remains the ſame, whatever permutations be made among them. Therefore, in changing a into b, and b into a, the ſecond member of the equation will be the ſame as before; hence we have this other equation,

$$\frac{\text{ſin.}\, a}{\text{ſin.}\, A} = \frac{\text{ſin.}\, b}{\text{ſin.}\, B} \ldots\ldots\ldots\ldots\ldots (B)$$

This is what is called the common analogy of the ſines, and it is obvious, that by changing a and c into A and C, we have in like manner $\frac{\text{ſin.}\, a}{\text{ſin.}\, A} = \frac{\text{ſin.}\, c}{\text{ſin.}\, C}$.

16. Let us reſume the equation (A) of No. 14, coſ. a = coſ. b coſ. c + ſin. b ſin. c coſ. A; by changing a into c, and A into C, we have in like manner coſ. c = coſ. a coſ. b + ſin. a ſin. b coſ. C; ſubſtituting this value of coſ. c in the firſt equation it becomes coſ. a = coſ. a coſ.^{2} b + ſin. a ſin. b coſ. b coſ. C

therefore coſ. A $= \dfrac{b^2+c^2-a^2}{2bc}$, and hence

$$\text{ſin. A} = \frac{\surd[4b^2c^2-(b^2+c^2-a^2)^2]}{2bc}.$$

Therefore, if in order to ſimplify, we put

$$d = \surd[4b^2c^2-(b^2+c^2-a^2)^2]$$

we have ſin. A $= \dfrac{d}{2bc}$, and hence $\dfrac{a}{\text{ſin. A}} = \dfrac{2abc}{d}$ which fraction expreſſes the ratio ſought.

If r be put for the radius of the circle which circumſcribes the triangle, and s for the ſurface, or area of the ſame triangle, we have $\dfrac{a}{\text{ſin. A}} = \dfrac{2abc}{d} = 2r = \dfrac{abc}{2s}$; therefore $r = \dfrac{abc}{d}$, and and $s = \dfrac{abc}{4r} = \dfrac{d}{4}$.

2. By developing the ſquare of $b^2+c^2-a^2$, and reducing, we have

$$d = \surd(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)$$

a formula in which the three ſides a, b, c are equally concerned, as they ought to be.

But this formula may be rendered more ſimple and more commodious for logarithmic calculation by reſolving it into factors. For

$$4b^2c^2-(b^2+c^2-a^2)^2 = (2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2),$$

and again reſolving each of theſe factors into two, we have

$$d = \surd[(a+b+c)(a-b+c)(a+b-c)(-a+b+c)].$$

Theſe formulas are already known, and I only give them here as an introduction to inveſtigations which are to follow.

3. Since in ſpherical triangles the ſines of the ſides are proportional to the ſines of the angles oppoſite to them, we may be deſirous to inveſtigate the conſtant ratio, and to diſcover if it depends, as in rectilineal triangles, upon the radius of the circumſcribed circle, or the area of the triangle.

Let us denote as before the three ſides of a ſpherical triangle by a, b, c, and the three angles oppoſite to thoſe ſides by A, B, C.

Then by a known theorem

$$\text{coſ. } a = \text{coſ. } b \text{ coſ. } c + \text{ſin. } b \text{ ſin. } c \text{ coſ. A};$$

therefore coſ. A $= \dfrac{\text{coſ. } a - \text{coſ. } b \text{ coſ. } c}{\text{ſin. } b \text{ ſin. } c}$; and hence

$$\text{ſin. A} = \frac{\surd[\text{ſin.}^2 b \text{ ſin.}^2 c - (\text{coſ. } a - \text{coſ. } b \text{ coſ. } c)^2]}{\text{ſin. } b \text{ ſin. } c}$$

Let us, to abreviate, put

$$f = \sqrt{[\text{ſin.}^2 b\ \text{ſin.}^2 c - (\text{coſ.}\ a - \text{coſ.}\ b\ \text{coſ.}\ c)^2]}$$

thus we have ſin. $A = \dfrac{f}{\text{ſin.}\ b\ \text{ſin.}\ c}$, and therefore

$$\frac{\text{ſin.}\ a}{\text{ſin.}\ A} = \frac{\text{ſin.}\ a\ \text{ſin.}\ b\ \text{ſin.}\ c}{f}$$

an expreſſion for the ratio ſought analogous to that which has been already found for rectilineal triangles (No. 1).

As the quantity f is expreſſed by a ſquare root, and may, therefore, have either the ſign plus, or minus prefixed to it, we remark, that relatively to ſpherical triangles, it ought to be always poſitive; for the ſides and the angles of every triangle being always leſs than two right angles, the ſines are neceſſarily always poſitive.

4. The radical quantity f is alſo ſuſceptible of reductions ſimilar to thoſe of No. 2. For by ſubſtituting inſtead of ſin.$^2\,b$ and ſin.$^2\,c$ their values $1 - \text{coſ.}^2 b$ and $1 - \text{coſ.}^2 c$; and afterwards reducing, we have

$$f = \sqrt{[1 - \text{coſ.}^2 a - \text{coſ.}^2 b - \text{coſ.}^2 c + 2\text{coſ.}\ a\ \text{coſ.}\ b\ \text{coſ.}\ c]}$$

where it appears that the three ſides a, b, c are alike concerned.

In like manner the quantity under the radical ſign may be reſolved into factors, for we have preſently

$$\begin{aligned} &\text{ſin.}^2 b\ \text{ſin.}^2 c - (\text{coſ.}\ a - \text{coſ.}\ b\ \text{coſ.}\ c)^2 \\ &= (\text{ſin.}\ b\ \text{ſin.}\ c + \text{coſ.}\ a - \text{coſ.}\ b\ \text{coſ.}\ c) \\ &\quad \times (\text{ſin.}\ b\ \text{ſin.}\ c - \text{coſ}\ a + \text{coſ.}\ b\ \text{coſ.}\ c) \\ &= [\text{coſ.}\ a - \text{coſ.}\ (b + c)]\ [\text{coſ.}\ (b - c) - \text{coſ.}\ a] \end{aligned}$$

but we have, in general $\text{coſ.}\ a - \text{coſ.}\ h = 2\ \text{ſin.}\ \dfrac{a+h}{2} \times \text{ſin.}\ \dfrac{h-a}{2}$

therefore, reſolving thus the two factors, we have

$$f = 2\sqrt{[\text{ſin.}\frac{a+b+c}{2} \times \text{ſin.}\frac{a-b+c}{2} \times \text{ſin.}\frac{a+b-c}{2} \times \text{ſin.}\frac{-a+b+c}{2}]}$$

an expreſſion very convenient for logarithmic calculation.

5. Let us now ſeek the radius of the circle circumſcribing the ſpherical triangle. It is evident that this circle muſt be a ſmall circle of the ſphere, and that it will alſo circumſcribe the rectilineal triangle formed by the chords of the arches a, b, c; but theſe chords being expreſſed by $2\ \text{ſin.}\ \frac{a}{2}$, $2\text{ſin.}\ \frac{b}{2}$, $2\text{ſin.}\ \frac{2}{c}$ it is only neceſſary to ſubſtitute theſe quantitities inſtead of a, b,

c in the expreſſion of the radius r (No. 2), that is to ſay, in $\frac{abc}{d}$.

Let us put R for the radius of the circumſcribed circle, and h for the value which d acquires by ſubſtituting for a, b, c as above; thus we have

$$R = \frac{8\,\text{ſin.}\,\frac{a}{2} \times \text{ſin.}\,\frac{b}{2} \times \text{ſin.}\,\frac{c}{2}}{h}.$$

But we have $h^2 = 32\,(\text{ſin.}\,\frac{a}{2}\;\text{ſin.}\,\frac{b}{2})^2 + 32\,(\text{ſin.}\,\frac{a}{2}\;\text{ſin.}\,\frac{c}{2})^2 + 32\,(\text{ſin.}\,\frac{b}{2}\;\text{ſin.}\,\frac{c}{2})^2 - 16\,(\text{ſin.}\,\frac{a}{2})^4 - 16(\text{sin.}\,\frac{b}{2})^4 - 16(\text{sin.}\,\frac{c}{2})^4$;

and ſubſtituting for $2(\text{sin.}\,\frac{a}{2})^2$, $2(\text{sin.}\,\frac{b}{2})^2$, $2(\text{sin.}\,\frac{c}{2})^2$ their values $1 - \text{coſ.}\,a$, $1 - \text{coſ.}\,b$, $1 - \text{coſ.}\,c$ we have after proper reductions

$$\begin{aligned} h^2 = {} & 12 - 8\,\text{coſ.}\,a - 8\,\text{coſ.}\,b - 8\,\text{coſ.}\,c \\ & + 8\,\text{coſ.}\,a\,\text{coſ.}\,b + 8\,\text{coſ.}\,a\,\text{coſ.}\,c + 8\,\text{coſ.}\,b\,\text{coſ.}\,c \\ & - 4\,\text{coſ.}^2\,a - 4\,\text{coſ.}^2\,b - 4\,\text{coſ.}^2\,c \end{aligned}$$

which expreſſion may be reduced to the following

$$4f^2 + 8(1 - \text{coſ.}\,a)\,(1 - \text{coſ.}\,b)\,(1 - \text{coſ.}\,c)$$

that is to ſay, to

$$4f^2 + 64\,(\text{ſin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2})^2$$

by ſubſtituting the value of f of No. 4. Thus we have

$$R = \frac{4\,\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2}}{\sqrt{[f^2 + 16(\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\;\text{sin.}\,\frac{c}{2})^2]}}.$$

6. Now by conſidering the radius of the ſphere which paſſes through the centre of the ſmall circle circumſcribing the triangle, it is evident that this radius is perpendicular to the plane of the circle, and that it meets the ſurface of the ſphere at a point which is the pole of the ſame circle.

Therefore, putting φ for the arch which meaſures the diſtance of the circle from its pole, we have evidently $R = \text{sin.}\,\varphi$, therefore

employed is alſo applicable to this caſe: for having determined the angle φ by equation (b) the equation (c) becomes

$$\text{coſ.}\,(b-\varphi)=\frac{\text{coſ.}\,a\ \text{coſ.}\,\varphi}{\text{coſ.}\,c}\ \ldots\ldots\ldots\ldots\ (d).$$

22. The equation (B) between two sides a and b, and the oppoſite angles A and B may ſerve 1ſt to determine A by a, b, B; and 2d to determine a by b, A, B.

1ſt, To determine A by a, b, B we have

$$\text{sin.}\,A=\frac{\text{sin.}\,a\ \text{sin.}\,B}{\text{sin.}\,b}$$

2d, To determine a by b, A, B we have

$$\text{sin.}\,a=\frac{\text{sin.}\,b\ \text{sin.}\,A}{\text{sin.}\,B}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (f)$$

Theſe formulas admit of the application of logarithms without further transformation.

23. The equation (C) between two sides a and b, and two angles A, C the firſt oppoſite, the ſecond adjacent to the side a, may ſerve 1ſt to determine A by a, b, C; 2d, to determine a by b, A, C; 3d to determine C by a, b, A; 4th to determine b by a, A, C.

1ſt, To determine A by a, b, C we have this equation,

$$\text{cot.}\,A=\frac{\text{cot.}\,a\ \text{sin.}\,b}{\text{sin.}\,C}-\text{cot.}\,C\ \text{coſ.}\,b;$$

and to reduce it into factors, we aſſume $\frac{\text{cot.}\,a}{\text{coſ.}\,C}=\text{cot.}\,\varphi$

or,

$$\text{tan.}\,a\ \text{coſ.}\,C=\text{tan.}\,\varphi\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (g)$$

therefore, $\text{cot.}\,a=\text{coſ.}\,C\ \text{cot.}\,\varphi$; and ſubſtituting this value we have $\text{cot.}\,A=\text{cot.}\,C\,(\text{cot.}\,\varphi\ \text{sin.}\,b-\text{coſ.}\,b)$

$=\frac{\text{cot.}\,C}{\text{sin.}\,\varphi}(\text{coſ.}\,\varphi\,\text{sin.}\,b-\text{sin.}\,\varphi\ \text{coſ.}\,b)=\frac{\text{cot.}\,C\ \text{sin.}\,(b-\varphi)}{\text{sin.}\,\varphi}$; theref.

$$\text{cot.}\,A=\frac{\text{cot.}\,C\ \text{sin.}\,(b-\varphi)}{\text{sin.}\,\varphi}\ \ldots\ldots\ldots\ldots\ldots\ (h)$$

This reduction comes alſo to the ſame thing, as the reſolving of the triangle into two right-angled triangles, by a perpendicular from the angle B upon the oppoſite side b, and the ſubſidiary angle φ is the ſegment of the side adjacent to the angle C.

2d, To determine a by b, A, C we have this equation,

$$\text{cot.}\,a=\frac{\text{cot.}\,A\ \text{sin.}\,C}{}+\text{cot.}\,b\ \ \text{coſ.}\,C$$

and

$$\frac{2\sin.\frac{a}{2}\sin.\frac{b}{2}\sin.\frac{c}{2}}{3\tan.\varphi}$$

and again, by ſubſtituting for tan. φ its value already found (No. 6), we have $\frac{\pounds}{6}$ for the ſolidity of the pyramid.

8. It remains for us to conſider the area of the ſpherical triangle itſelf formed by the arcs a, b, c. There is a beautiful and known theorem, according to which, the area of a ſpherical triangle is to the whole ſurface of the ſphere, as the exceſs of the three angles of the triangle above two right angles to eight right angles. This theorem is commonly attributed to Albert Girard, who gave it in ſubſtance, in a work entitled *Invention Nouvelle en Algebre*, printed at Amſterdam in 1629; but as the proof which he gives is not rigorous, and cannot even be considered as an induction, the theorem ought rather to be attributed to Cavallerius, who gave it in the *Directericum generale uranometricum*, printed at Bologna in 1632, with the elegant demonſtration firſt publiſhed by Wallis, and since inſerted in moſt works upon trigonometry.

Let Σ denote the exceſs of the three angles of the triangle above two right angles, then, retaining the notation hitherto employed, and putting D for a right angle,

$$\Sigma = A + B + C - 2D.$$

Thus the area of the triangle of which the sides are a, b, c and the angles A, B, C ſhall be the $\frac{\Sigma}{8D}$ part of the whole ſurface of the ſphere; and if this ſurface be considered as equal to 8D, we may take Σ for the area of the triangle.

9. If we ſuppoſe the sides b and c, which contain the angle A, to be produced till each of them be a quadrant, the angles B, C will then become right angles, and the side a will become equal to the oppoſite angle A; then the area of the iſoſceles right angled triangle becomes A; therefore, if from it we ſubtract the former triangle, of which the sides about the angle A are b and c, we ſhall have a ſpherical quadrilateral, the baſe of which is A, and the sides perpendicular to that baſe $D - b$ and $D - c$, and the area of the quadrilateral ſhall be expreſſed simply by $B + C - 2D$.

But by the known analogies of Napier, we have in every ſpherical triangle this equation, which will be demonſtrated farther on,

$$\tan.\frac{B+C}{2} = \frac{\text{coſ.}\,\frac{b-c}{2}}{\text{coſ.}\,\frac{b+c}{2}}\cot.\frac{A}{2}.$$

Therefore, if σ denote the ſurface of the quadrilateral we are conſidering, we ſhall have

$$\tan.\frac{\sigma}{2} = \cot.\frac{B+C}{2} = \frac{1}{\tan.\frac{B+C}{2}};\ \text{therefore,}$$

$$\tan.\frac{\sigma}{2} = \frac{\text{coſ.}\,\frac{b+c}{2}}{\text{coſ.}\,\frac{b-c}{2}}\tan.\frac{A}{2};$$

and if we put β and γ for the two ſides of the quadrilateral perpendicular to the baſe A, ſo that $\beta = D - b$ and $\gamma = D - c$ we have in order to determine the area σ this formula

$$\tan.\frac{\sigma}{2} = \frac{\sin.\frac{\beta+\gamma}{2}}{\text{coſ.}\,\frac{\beta-\gamma}{2}}\tan.\frac{A}{2}.$$

This formula correſponds to the known formula $\sigma = \frac{\beta+\gamma}{2}A$ for a rectilineal quadrilateral of which A is the baſe, β and γ the two vertical ſides, and σ the area; and as the latter is of the greateſt utility in the menſuration of plane ſurfaces bounded by ſtraight lines, ſo the former is equally uſeful in the menſuration of ſpherical ſurfaces bounded by arches of great circles. Thus it may be employed with much advantage to determine the extent of a country, when the latitudes and longitudes of ſeveral points in its circumference are known; for by connecting thoſe points by arches of great circles, we have a ſpherical polygon, the area of which may be eaſily found, by reſolving it into quadrilaterals formed by circles of latitudes and arches of the equator intercepted between thoſe circles.

10. But if it was required to expreſs the area Σ by the three ſides *a*, *b*, *c* of the ſpherical triangle, it is only neceſſary to conſider, that ſince $\Sigma = A + B + C - 2D$ we have

$$\cot.\frac{\Sigma}{2} = -\tan.\frac{A+B+C}{2}$$

$$= -\frac{\tan.\frac{A}{2} + \tan.\frac{B+C}{2}}{1 - \tan.\frac{A}{2}\tan.\frac{B+C}{2}}.$$

$$\frac{2\sin.\frac{a}{2}\sin.\frac{b}{2}\sin.\frac{c}{2}}{3\tan.\,\varphi}$$

and again, by ſubſtituting for tan. φ its value already found (No. 6), we have $\frac{f}{6}$ for the ſolidity of the pyramid.

8. It remains for us to conſider the area of the ſpherical triangle itſelf formed by the arcs a, b, c. There is a beautiful and known theorem, according to which, the area of a ſpherical triangle is to the whole ſurface of the ſphere, as the exceſs of the three angles of the triangle above two right angles to eight right angles. This theorem is commonly attributed to Albert Girard, who gave it in ſubſtance, in a work entitled *Invention Nouvelle en Algebre*, printed at Amſterdam in 1629; but as the proof which he gives is not rigorous, and cannot even be conſidered as an induction, the theorem ought rather to be attributed to Cavallerius, who gave it in the *Directericum generale uranometricum*, printed at Bologna in 1632, with the elegant demonſtration firſt publiſhed by Wallis, and ſince inſerted in moſt works upon trigonometry.

Let Σ denote the exceſs of the three angles of the triangle above two right angles, then, retaining the notation hitherto employed, and putting D for a right angle,

$$\Sigma = A + B + C - 2D.$$

Thus the area of the triangle of which the ſides are a, b, c and the angles A, B, C ſhall be the $\frac{\Sigma}{8D}$ part of the whole ſurface of the ſphere; and if this ſurface be conſidered as equal to 8D, we may take Σ for the area of the triangle.

9. If we ſuppoſe the ſides b and c, which contain the angle A, to be produced till each of them be a quadrant, the angles B, C will then become right angles, and the ſide a will become equal to the oppoſite angle A; then the area of the iſoſceles right angled triangle becomes A; therefore, if from it we ſubtract the former triangle, of which the ſides about the angle A are b and c, we ſhall have a ſpherical quadrilateral, the baſe of which is A, and the ſides perpendicular to that baſe $D - b$ and $D - c$, and the area of the quadrilateral ſhall be expreſſed ſimply by $B + C - 2D$.

But by the known analogies of Napier, we have in every ſpherical triangle this equation, which will be demonſtrated farther on,

$$\tan.\frac{B+C}{2}=\frac{\operatorname{cof}.\frac{b-c}{2}}{\operatorname{cof}.\frac{b+c}{2}}\cot.\frac{A}{2}.$$

Therefore, if σ denote the ſurface of the quadrilateral we are conſidering, we ſhall have

$$\tan.\frac{\sigma}{2}=\cot.\frac{B+C}{2}=\frac{1}{\tan.\frac{B+C}{2}};\text{ therefore,}$$

$$\tan.\frac{\sigma}{2}=\frac{\operatorname{cof}.\frac{b+c}{2}}{\operatorname{cof}.\frac{b-c}{2}}\tan.\frac{A}{2};$$

and if we put β and γ for the two ſides of the quadrilateral perpendicular to the baſe A, ſo that $\beta=D-b$ and $\gamma=D-c$ we have in order to determine the area σ this formula

$$\tan.\frac{\sigma}{2}=\frac{\sin.\frac{\beta+\gamma}{2}}{\operatorname{cof}.\frac{\beta-\gamma}{2}}\tan.\frac{A}{2}.$$

This formula correſponds to the known formula $\sigma=\frac{\beta+\gamma}{2}A$ for a rectilineal quadrilateral of which A is the baſe, β and γ the two vertical ſides, and σ the area; and as the latter is of the greateſt utility in the menſuration of plane ſurfaces bounded by ſtraight lines, ſo the former is equally uſeful in the menſuration of ſpherical ſurfaces bounded by arches of great circles. Thus it may be employed with much advantage to determine the extent of a country, when the latitudes and longitudes of ſeveral points in its circumference are known; for by connecting thoſe points by arches of great circles, we have a ſpherical polygon, the area of which may be eaſily found, by reſolving it into quadrilaterals formed by circles of latitudes and arches of the equator intercepted between thoſe circles.

10. But if it was required to expreſs the area Σ by the three ſides a, b, c of the ſpherical triangle, it is only neceſſary to conſider, that ſince $\Sigma=A+B+C-2D$ we have

$$\cot.\frac{\Sigma}{2}=-\tan.\frac{A+B+C}{2}$$

$$=-\frac{\tan.\frac{A}{2}+\tan.\frac{B+C}{2}}{1-\tan.\frac{A}{2}\tan.\frac{B+C}{2}}.$$

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therefore $\text{coſ. A} = \dfrac{b^2 + c^2 - a^2}{2bc}$, and hence

$$\text{ſin. A} = \frac{\sqrt{[4b^2c^2 - (b^2 + c^2 - a^2)^2]}}{2bc}.$$

Therefore, if in order to ſimplify, we put

$$d = \sqrt{[4b^2c^2 - (b^2 + c^2 - a^2)^2]}$$

we have $\text{ſin. A} = \dfrac{d}{2bc}$, and hence $\dfrac{a}{\text{ſin. A}} = \dfrac{2abc}{d}$ which fraction expreſſes the ratio ſought.

If r be put for the radius of the circle which circumſcribes the triangle, and s for the ſurface, or area of the ſame triangle, we have $\dfrac{a}{\text{ſin. A}} = \dfrac{2abc}{d} = 2r = \dfrac{abc}{2s}$; therefore $r = \dfrac{abc}{d}$, and and $s = \dfrac{abc}{4r} = \dfrac{d}{4}$.

2. By developing the ſquare of $b^2 + c^2 - a^2$, and reducing, we have

$$d = \sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)}$$

a formula in which the three ſides a, b, c are equally concerned, as they ought to be.

But this formula may be rendered more ſimple and more commodious for logarithmic calculation by reſolving it into factors. For

$$4b^2c^2 - (b^2 + c^2 - a^2)^2 = (2bc + b^2 + c^2 - a^2)\,(2bc - b^2 - c^2 + a^2),$$

and again reſolving each of theſe factors into two, we have

$$d = \sqrt{[(a + b + c)\,(a - b + c)\,(a + b - c)\,(-a + b + c)]}.$$

Theſe formulas are already known, and I only give them here as an introduction to inveſtigations which are to follow.

3. Since in ſpherical triangles the ſines of the ſides are proportional to the ſines of the angles oppoſite to them, we may be deſirous to inveſtigate the conſtant ratio, and to diſcover if it depends, as in rectilineal triangles, upon the radius of the circumſcribed circle, or the area of the triangle.

Let us denote as before the three ſides of a ſpherical triangle by a, b, c, and the three angles oppoſite to thoſe ſides by A, B, C.

Then by a known theorem

$$\text{coſ. } a = \text{coſ. } b \text{ coſ. } c + \text{ſin. } b \text{ ſin. } c \text{ coſ. A};$$

therefore $\text{coſ. A} = \dfrac{\text{coſ. } a - \text{coſ. } b \text{ coſ. } c}{\text{ſin. } b \text{ ſin. } c}$; and hence

$$\text{ſin. A} = \frac{\sqrt{[\text{ſin.}^2 b \text{ ſin.}^2 c - (\text{coſ. } a - \text{coſ. } b \text{ coſ. } c)^2]}}{\text{ſin. } b \text{ ſin. } c}$$

Let us, to abreviate, put

$$f = \sqrt{[\text{ſin.}^2 b\,\text{ſin.}^2 c - (\text{coſ.}\,a - \text{coſ.}\,b\,\text{coſ.}\,c)^2]}$$

thus we have ſin. A $= \frac{f}{\text{ſin.}\,b\,\text{ſin.}\,c}$, and therefore

$$\frac{\text{ſin.}\,a}{\text{ſin.}\,A} = \frac{\text{ſin.}\,a\,\text{ſin.}\,b\,\text{ſin.}\,c}{f}$$

an expreſſion for the ratio ſought analogous to that which has been already found for rectilineal triangles (No. 1).

As the quantity f is expreſſed by a ſquare root, and may, therefore, have either the ſign plus, or minus prefixed to it, we remark, that relatively to ſpherical triangles, it ought to be always poſitive; for the ſides and the angles of every triangle being always leſs than two right angles, the ſines are neceſſarily always poſitive.

4. The radical quantity f is alſo ſuſceptible of reductions ſimilar to thoſe of No. 2. For by ſubſtituting inſtead of ſin.$^2\,b$ and ſin.$^2\,c$ their values $1 - \text{coſ.}^2\,b$ and $1 - \text{coſ.}^2\,c$; and afterwards reducing, we have

$$f = \sqrt{[1 - \text{coſ.}^2 a - \text{coſ.}^2 b - \text{coſ.}^2 c + 2\text{coſ.}\,a\,\text{coſ.}\,b\,\text{coſ.}\,c]}$$

where it appears that the three ſides a, b, c are alike concerned.

In like manner the quantity under the radical ſign may be reſolved into factors, for we have preſently

$$\begin{aligned}
&\text{ſin.}^2 b\,\text{ſin.}^2 c - (\text{coſ.}\,a - \text{coſ.}\,b\,\text{coſ.}\,c)^2\\
&= (\text{ſin.}\,b\,\text{ſin.}\,c + \text{coſ.}\,a - \text{coſ.}\,b\,\text{coſ.}\,c)\\
&\times (\text{ſin.}\,b\,\text{ſin.}\,c - \text{coſ}\,a + \text{coſ.}\,b\,\text{coſ.}\,c)\\
&= [\text{coſ.}\,a - \text{coſ.}\,(b + c)]\,[\text{coſ.}\,(b - c) - \text{coſ.}\,a]
\end{aligned}$$

but we have, in general $\text{coſ.}\,a - \text{coſ.}\,h = 2\,\text{ſin.}\,\frac{a+h}{2} \times \text{ſin.}\,\frac{h-a}{2}$

therefore, reſolving thus the two factors, we have

$$f = 2\sqrt{[\text{ſin.}\,\tfrac{a+b+c}{2} \times \text{ſin.}\,\tfrac{a-b+c}{2} \times \text{ſin.}\,\tfrac{a+b-c}{2} \times \text{ſin.}\,\tfrac{-a+b+c}{2}]}$$

an expreſſion very convenient for logarithmic calculation.

5. Let us now ſeek the radius of the circle circumſcribing the ſpherical triangle. It is evident that this circle muſt be a ſmall circle of the ſphere, and that it will alſo circumſcribe the rectilineal triangle formed by the chords of the arches a, b, c; but theſe chords being expreſſed by $2\,\text{ſin.}\,\frac{a}{2}$, $2\text{ſin.}\,\frac{b}{2}$, $2\text{ſin.}\,\frac{2}{c}$. it is only neceſſary to ſubſtitute theſe quantitities inſtead of a, b,

c in the expreſſion of the radius r (No. 2), that is to ſay, in $\frac{abc}{d}$.

Let us put R for the radius of the circumſcribed circle, and h for the value which d acquires by ſubſtituting for a, b, c as above; thus we have

$$R = \frac{8\,\text{ſin.}\,\frac{a}{2} \times \text{ſin.}\,\frac{b}{2} \times \text{ſin.}\,\frac{c}{2}}{h}.$$

But we have $h^2 = 32\,(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{b}{2})^2 + 32\,(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{c}{2})^2 + 32\,(\text{ſin.}\,\frac{b}{2}\,\text{ſin.}\,\frac{c}{2})^2 - 16\,(\text{ſin.}\,\frac{a}{2})^4 - 16(\text{sin.}\,\frac{b}{2})^4 - 16(\text{sin.}\,\frac{c}{2})^4$;

and ſubſtituting for $2(\text{sin.}\,\frac{a}{2})^2$, $2(\text{sin.}\,\frac{b}{2})^2$, $2(\text{sin.}\,\frac{c}{2})^2$ their values $1 - \text{coſ.}\,a$, $1 - \text{coſ.}\,b$, $1 - \text{coſ.}\,c$ we have after proper reductions

$$\begin{aligned} h^2 = {} & 12 - 8\,\text{coſ.}\,a - 8\,\text{coſ.}\,b - 8\,\text{coſ.}\,c \\ & + 8\,\text{coſ.}\,a\,\text{coſ.}\,b + 8\,\text{coſ.}\,a\,\text{coſ.}\,c + 8\,\text{coſ.}\,b\,\text{coſ.}\,c \\ & - 4\,\text{coſ.}^2\,a - 4\,\text{coſ.}^2\,b - 4\,\text{coſ.}^2\,c \end{aligned}$$

which expreſſion may be reduced to the following

$$4f^2 + 8(1 - \text{coſ.}\,a)\,(1 - \text{coſ.}\,b)\,(1 - \text{coſ.}\,c)$$

that is to ſay, to

$$4f^2 + 64\,(\text{ſin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2})^2$$

by ſubſtituting the value of f of No. 4. Thus we have

$$R = \frac{4\,\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2}}{\sqrt{[f^2 + 16(\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2})^2]}}.$$

6. Now by conſidering the radius of the ſphere which paſſes through the centre of the ſmall circle circumſcribing the triangle, it is evident that this radius is perpendicular to the plane of the circle, and that it meets the ſurface of the ſphere at a point which is the pole of the ſame circle.

Therefore, putting φ for the arch which meaſures the diſtance of the circle from its pole, we have evidently $R = \text{sin.}\,\varphi$, therefore

sin.

$$\text{sin.}\,\varphi = \frac{4\,\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2}}{\sqrt{[f^2 + 16\,(\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2})^2]}}.$$

from whence we derive

$$\text{coſ.}\,\varphi = \frac{f}{\sqrt{[f^2 + 16(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{b}{2}\,\text{ſin.}\,\frac{c}{2})^2]}},$$

and hence

$$\text{tan.}\,\varphi = \frac{4\,\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2}}{f}.$$

Wherefore, ſeeing that $\text{sin.}\,a = 2\,\text{sin.}\,\frac{a}{2}\,\text{coſ.}\,\frac{a}{2}$, and ſo on with reſpect to $\text{sin.}\,b$ and $\text{sin.}\,c$, we have (No. 3).

$$\frac{\text{sin.}\,a}{\text{sin.}\,A} = 2\,\text{tan.}\,\varphi\,\text{coſ.}\,\frac{a}{2}\,\text{coſ.}\,\frac{b}{2}\,\text{coſ.}\,\frac{c}{2}.$$

7. If it were required to find the area of the rectilineal triangle formed by the chords of the arches a, b, c and inſcribed in the circle we have juſt now been considering; by putting S for that area, it would only be neceſſary to change, in the formula $s = \frac{abc}{4r}$ of No. 1, s into S, r into R, and a, b, c into $2\,\text{sin.}\,\frac{a}{2}$; $2\,\text{sin.}\,\frac{b}{2}$, $2\,\text{sin.}\,\frac{c}{2}$ thus we ſhould have immediately

$$S = \frac{2\,\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2}}{R}$$

or, becauſe $R = \text{sin.}\,\varphi$ (No. 6)

$$S = \frac{2\,\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2}}{\text{sin.}\,\varphi}.$$

If now we consider the triangular pyramid which has this triangle for its baſe, and of which the vertex is at the centre of the ſphere, it is evident that the height of the pyramid is $\text{coſ.}\,\varphi$, therefore its ſolidity is $\frac{S\,\text{coſ.}\,\varphi}{3}$, or, ſubſtituting for S the value we have already found,

$$\frac{2\sin.\frac{a}{2}\sin.\frac{b}{2}\sin.\frac{c}{2}}{3\tan.\varphi}$$

and again, by ſubſtituting for tan. φ its value already found (No. 6), we have $\frac{\pounds}{6}$ for the ſolidity of the pyramid.

8. It remains for us to conſider the area of the ſpherical triangle itſelf formed by the arcs a, b, c. There is a beautiful and known theorem, according to which, the area of a ſpherical triangle is to the whole ſurface of the ſphere, as the exceſs of the three angles of the triangle above two right angles to eight right angles. This theorem is commonly attributed to Albert Girard, who gave it in ſubſtance, in a work entitled *Invention Nouvelle en Algebre*, printed at Amſterdam in 1629; but as the proof which he gives is not rigorous, and cannot even be conſidered as an induction, the theorem ought rather to be attributed to Cavallerius, who gave it in the *Directericum generale uranometricum*, printed at Bologna in 1632, with the elegant demonſtration firſt publiſhed by Wallis, and ſince inſerted in moſt works upon trigonometry.

Let Σ denote the exceſs of the three angles of the triangle above two right angles, then, retaining the notation hitherto employed, and putting D for a right angle,

$$\Sigma = A + B + C - 2D.$$

Thus the area of the triangle of which the ſides are a, b, c and the angles A, B, C ſhall be the $\frac{\Sigma}{8D}$ part of the whole ſurface of the ſphere; and if this ſurface be conſidered as equal to 8D, we may take Σ for the area of the triangle.

9. If we ſuppoſe the ſides b and c, which contain the angle A, to be produced till each of them be a quadrant, the angles B, C will then become right angles, and the ſide a will become equal to the oppoſite angle A; then the area of the iſoſceles right angled triangle becomes A; therefore, if from it we ſubtract the former triangle, of which the ſides about the angle A are b and c, we ſhall have a ſpherical quadrilateral, the baſe of which is A, and the ſides perpendicular to that baſe $D - b$ and $D - c$, and the area of the quadrilateral ſhall be expreſſed ſimply by $B + C - 2D$.

But by the known analogies of Napier, we have in every ſpherical triangle this equation, which will be demonſtrated farther on,

tan.

Take $\sqrt{\frac{m}{1-m}} : 1 ::$ hyperbolic ſector : circular ſector VCY, and make $Cp = CT$; then p is a point in the trajectory.

From this conſtruction it will eaſily appear, that the number of revolutions which the body muſt make before it arrives at the centre will be infinite, becauſe the area CVA increaſes without limit.

Or the trajectory may be conſtructed in the following manner. Deſcribe the ſemicircle SVD, (fig. 155, pl. 8.) with the radius $CV = r$; draw AB perpendicular to CD, and ſuppoſe it equal to y, and take the arc VQ = the difference of the hyperbolic logarithms of AB and BD multiplied into $r\sqrt{\frac{m}{1-m}}$, and take $Cp = AB$; then is p a point in the trajectory, as will be evident from the equation $z = r\sqrt{\left(\frac{m}{1-m}\right)} \times \log. \frac{y}{r-\sqrt{(r^2-y^2)}}$.

Becauſe the ratio of AB to BD is infinitely great when AD is evaneſcent, the number of revolutions before the body arrives at the centre, muſt be infinite.

Cor. 4. If $n = 4$, then $p = \dfrac{\sqrt{\left(\frac{m}{m-1}\right)} \times Py^{\frac{3}{2}}}{\sqrt{\left(y^3 + \frac{r^3}{m-1}\right)}}$

$\dot{z} = \pm \dfrac{\sqrt{\left(\frac{m}{m-1}\right)} \times rPy^{\frac{1}{2}}\dot{y}}{y\sqrt{\left(y^3 - \frac{m}{m-1} \times P^2y + \frac{r^3}{m-1}\right)}}$, and $y^3 - \frac{m}{m-1} P^2 y + \frac{r^3}{m-1} = 0$, is the equation for determining the apſides; where m may be greater than, equal to, or leſs than unity. The ratios of m to unity, and of P to r being given, it will be eaſy to determine when all the roots of this laſt equation are real, and when two of them are impoſſible. If the body be projected from an apſe, or $P = r$, then all the roots are real; two being poſitive and one negative, which belongs to a part of the curve having its concavity turned from the centre of force.

If $n = 5$, then $p = \dfrac{\sqrt{\left(\frac{m}{m-1}\right)} \times Py^2}{\sqrt{\left(y^4 + \frac{r^4}{m-1}\right)}} = \dfrac{\sqrt{m} \times Py^2}{\sqrt{\{(m-1)y^4 + r^4\}}}$,

If we ſubſtitute, in place of tan. $\frac{B+C}{2}$ its value found above (preceding No.), we have

$$\text{cot.}\ \frac{\Sigma}{2} = -\frac{\text{tan.}\ \frac{A}{2}\ \text{coſ.}\ \frac{b+c}{2} + \text{cot.}\ \frac{A}{2}\ \text{coſ.}\ \frac{b-c}{2}}{\text{coſ.}\ \frac{b+c}{2} - \text{coſ.}\ \frac{b-c}{2}},$$

a formula, which is eaſily transformed to this other

$$\text{cot.}\ \frac{\Sigma}{2} = \frac{\text{coſ.}\ \frac{b}{2}\ \text{coſ.}\ \frac{c}{2} + \text{ſin.}\ \frac{b}{2}\ \text{ſin.}\ \frac{c}{2}\ \text{coſ.}\ A}{\text{ſin.}\ \frac{b}{2}\ \text{ſin.}\ \frac{c}{2}\ \text{ſin.}\ A}.$$

If now we ſubſtitute in this formula the values ſin. A, and coſ. A of No. 3, we ſhall have, by dividing the numerator and denominator by ſin. $\frac{b}{2}$ ſin. $\frac{c}{2}$

$$\text{cot.}\ \frac{\Sigma}{2} = \frac{4(\text{coſ.}\ \frac{b}{2}\ \text{coſ.}\ \frac{c}{2})^2 + \text{coſ.}\ a - \text{coſ.}\ b\ \text{coſ.}\ c}{f};$$

But $2(\text{coſ.}\ \frac{b}{2})^2 = 1 + \text{coſ.}\ b$, and $2(\text{coſ.}\ \frac{c}{2})^2 = 1 + \text{coſ.}\ c$, therefore, making the neceſſary ſubſtitutions and inverting the fraction,

$$\text{tan.}\ \frac{\Sigma}{2} = \frac{f}{1 + \text{coſ.}\ a + \text{coſ.}\ b + \text{coſ.}\ c}$$

a formula of the moſt ſimple form to determine the area of a ſpherical triangle by means of its three ſides a, b, c.

11. We have ſeen (No. 7) that $\frac{f}{6}$ is the ſolidity of the triangular pyramid formed by the three radii of the ſphere which terminate in the angles of a ſpherical triangle. Let us now conſider a triangular pyramid formed by the ſame radii produced at pleaſure in ſuch a manner that they become p, q, r; and that a, b, c are the arches, or angles contained by thoſe ſtraight lines. To have the ſolidity of this pyramid, it may be conſidered as lying on one of its faces; for inſtance, that which has for its ſides the lines p, q, and to have a perpendicular P drawn from the other ſide r upon the plane of that face. It is immediately obvious, that if a is the angle contained by p and q, the area of the face which we have conſidered as the baſe of the

pyramid will be $\frac{pq \text{ ſin. } a}{2}$, therefore, the ſolidity of the pyramid will be $\frac{Ppq \text{ ſin. } a}{6}$. But if we call θ the angle which the line r makes with the plane paſſing through p and q, it is evident that we have $P = r$ ſin. θ, therefore, the ſolidity required will be $\frac{pqr \text{ ſin. } a \text{ ſin. } \theta}{6}$.

The angle θ is no other than the arch let fall perpendicularly from the angle A of the ſpherical triangle upon the oppoſite ſide a; we may, therefore, expreſs ſin. θ by the ſines or coſines of a, b, c the ſides of the triangle; but for our purpoſe it is ſufficient to conſider that this value, as well as that of ſin. a, being independent of the lines p, q, r, if we make $p = 1$, $q = 1$, $r = 1$ we ſhall have the caſe of the particular pyramid we propoſe to conſider; and of which the ſolidity is $\frac{f}{6}$. Hence it follows, that we have ſin. a ſin. $\theta = f$, and therefore $\frac{pqrf}{6}$ for the ſolidity of the triangular pyramid in which the three ſides, or edges that form any one of the ſolid angles are p, q, r, and the angles contained by thoſe three ſides are A, B, C.

This expreſſion, for the ſolidity of any triangular pyramid, by means of its three ſides and contained angles, is evidently very ſimple and commodious for calculation, more eſpecially if we employ the value of f, as expreſſed by factors, in No. 4; and it may be very uſeful to determine the ſolid contents of all bodies bounded by planes, ſeeing that they may always be reſolved into triangular pyramids, as a polygon is reſolved into triangles.

12. We may add, that ſince we have found ſin. a ſin. $\theta = f$, we ſhall have ſin. $\theta = \frac{f}{\text{ſin. } a}$. We may thus determine by this formula the perpendicular θ in a ſpherical triangle, of which a is the baſe, and b, c the two ſides.

13. I have reſolved the preceding problems, only to ſhew the origin and uſe of ſome remarkable formulas, and more eſpecially of that function which I have denoted by f, and which deſerves in a particular manner the attention of analyſts, on account of its different applications. I now proceed to general conſiderations relating to ſpherical trigonometry, viewed analytically.

Analytical reſolutions of ſpherical triangles were at firſt only

ſimple applications of algebra to geometrical conſtructions. It was afterwards thought ſufficient to eſtabliſh ſome fundamental propoſitions by the aid of geometry, and all the formulas of ſpherical trigonometry have been deduced from equations derived from thoſe propoſitions. The moſt elegant production of this kind is a Memoir of Euler's, entitled *Trigonometria ſpherica univerſa ex primis principiis derivata*, and printed in the Petersburgh Acts for the year 1779, in which there is found a complete ſyſtem of trigonometrical formulas, founded ſolely upon three equations. But may not the ſyſtem be yet more ſimplified by reducing it to one fundamental equation?

Such a reduction will complete the analytical theory of ſpherical triangles; for, in Analyſis, perfection conſiſts in employing the ſmalleſt poſſible number of principles, and in deducing from theſe all the truths they contain by the powers of that method alone: on the contrary, in the ſynthetic method of lines, it conſiſts in demonſtrating ſeparately every propoſition in the moſt ſimple manner, by means of other propoſitions previouſly demonſtrated.

The late de Gua had conceived the idea of making all ſpherical trigonometry to depend upon a ſingle general property of ſpherical triangles; but the Memoir he has given upon the ſubject, in the volume of the Academy of Sciences for 1783, contains calculations ſo complicated, that they ſeem better adapted to ſhew the inconvenience of his method than to cauſe it to be adopted.

I here propoſe to myſelf the ſame object, and I proceed to give a conciſe view of all the formulas of ſpherical trigonometry, deducing them by ſimple transformations from a ſingle equation, given by the nature of ſpherical triangles.

Let us aſſume, as De Gua has done, the equation (No. 3.)

$$\text{coſ.}\ a = \text{coſ.}\ b\ \text{coſ.}\ c + \text{ſin.}\ b\ \text{ſin.}\ c\ \text{coſ.}\ A$$

in which a, b, c are the three ſides or arches of the triangle, and A the angle oppoſite to the ſide a.

This equation is eaſily demonſtrated by the ſimple conſideration of two rectilineal triangles formed, the one by the two tangents to the arches b, c, and by the ſtraight line which joins the extremities of theſe tangents, and the other by the ſame line and the two ſecants of the ſame arches; for it is obvious, that the two tangents form between them the angle A which is contained by the arches b and c, and that the two ſecants form between them the angle a which is the ſide of the ſpherical triangle oppoſite to the angle A. Thus, calling h the ſide common to theſe two triangles, we have immediately, by a known theorem concerning rectilineal triangles, this equation,

$$h^2 = \text{tan.}^2\, b + \text{tan.}^2\, c - 2\text{tan.}\ b\ \text{tan.}\ c\ \text{coſ.}\ A$$

for the former triangle, and this other equation,

$$h^2 = \text{ſec.}^2\, b + \text{ſec.}^2\, c - 2\text{ſec.}\ b\ \text{ſec.}\ c\ \text{coſ.}\ a$$

for the latter triangle.

Hence we find

$$\tan.^2 b + \tan.^2 c - 2 \tan. b \tan. c \text{ coſ. } A$$
$$= \text{ſec.}^2 b + \text{ſec.}^2 c - 2 \text{ ſec. } b \text{ ſec. } c \text{ coſ. } a.$$

But we have evidently

$$\text{ſec.}^2 b - \tan.^2 b = 1, \text{ alſo ſec.}^2 c - \tan.^2 c = 1,$$

therefore, the equation becomes

$$\text{ſec. } b \text{ ſec. } c \text{ coſ. } a = 1 + \tan. b \tan. c \text{ coſ. } A,$$

hence, ſubſtituting for ſec. b, ſec. c, tan. b, tan. c, their values $\frac{1}{\text{coſ. } b}$, $\frac{1}{\text{coſ. } c}$, $\frac{\sin. b}{\text{coſ.} b}$, $\frac{\sin. c}{\text{coſ.} c}$, we have the fundamental equation,

$$\text{coſ. } a = \text{coſ. } b \text{ coſ. } c + \sin. b \sin. c \text{ coſ. } A \ldots\ldots\ldots\ldots (A)$$

As, by hypotheſis, the only condition ſuppoſed to have place among the four quantities a, b, c, A is that a, b, c are the three ſides of a triangle, and A the angle oppoſite to the ſide a; it follows, that by putting B and C for the angles oppoſite to the ſides b and c, we ſhall have ſimilar equations relatively to thoſe angles, by ſimply changing A into B, or into C, provided that at the ſame time we change a into b, or into c.

15. Now if we deduce the value of coſ. A from the preceding equations, and thence that of sin. A we ſhall have, as has been already found in No. 3,

$$\frac{\sin. a}{\sin. A} = \frac{\sin. a \sin. b \sin. c}{f}$$

where the quantity f is a function of a, b, c into which theſe three quantities enter similarly, ſo that it remains the ſame, whatever permutations be made among them. Therefore, in changing a into b, and b into a, the ſecond member of the equation will be the ſame as before; hence we have this other equation,

$$\frac{\sin. a}{\sin. A} = \frac{\sin. b}{\sin. B} \ldots\ldots\ldots\ldots\ldots (B)$$

This is what is called the common analogy of the sines, and it is obvious, that by changing a and c into A and C, we have in like manner $\frac{\sin. a}{\sin. A} = \frac{\sin. c}{\sin. C}$.

16. Let us reſume the equation (A) of No. 14, coſ. $a =$ coſ. b coſ. $c +$ sin. b sin. c coſ. A; by changing a into c, and A into C, we have in like manner coſ. $c =$ coſ. a coſ. $b +$ sin. a sin. b coſ. C; ſubſtituting this value of coſ. c in the firſt equation it becomes coſ. $a =$ coſ. a coſ.^{2} b + sin. a sin. b coſ. b coſ.

c in the expreſſion of the radius r (No. 2), that is to ſay, in $\frac{abc}{d}$.

Let us put R for the radius of the circumſcribed circle, and h for the value which d acquires by ſubſtituting for a, b, c as above; thus we have

$$R = \frac{8\,\text{ſin.}\,\frac{a}{2} \times \text{ſin.}\,\frac{b}{2} \times \text{ſin.}\,\frac{c}{2}}{h}.$$

But we have $h^2 = 32\,(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{b}{2})^2 + 32\,(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{c}{2})^2 +$

$32\,(\text{ſin.}\,\frac{b}{2}\,\text{ſin.}\,\frac{c}{2})^2 - 16\,(\text{ſin.}\,\frac{a}{2})^4 - 16(\text{sin.}\,\frac{b}{2})^4 - 16(\text{sin.}\,\frac{c}{2})^4$;

and ſubſtituting for $2(\text{sin.}\,\frac{a}{2})^2$, $2(\text{sin.}\,\frac{b}{2})^2$, $2(\text{sin.}\,\frac{c}{2})^2$ their values $1 - \text{coſ.}\,a$, $1 - \text{coſ.}\,b$, $1 - \text{coſ.}\,c$ we have after proper reductions

$$\begin{aligned} h^2 = {} & 12 - 8\,\text{coſ.}\,a - 8\,\text{coſ.}\,b - 8\,\text{coſ.}\,c \\ & + 8\,\text{coſ.}\,a\,\text{coſ.}\,b + 8\,\text{coſ.}\,a\,\text{coſ.}\,c + 8\,\text{coſ.}\,b\,\text{coſ.}\,c \\ & - 4\,\text{coſ.}^2\,a - 4\,\text{coſ.}^2\,b - 4\,\text{coſ.}^2\,c \end{aligned}$$

which expreſſion may be reduced to the following

$$4f^2 + 8(1 - \text{coſ.}\,a)\,(1 - \text{coſ.}\,b)\,(1 - \text{coſ.}\,c)$$

that is to ſay, to

$$4f^2 + 64\,(\text{ſin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2})^2$$

by ſubſtituting the value of f of No. 4. Thus we have

$$R = \frac{4\,\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2}}{\sqrt{[f^2 + 16(\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2})^2]}}.$$

6. Now by conſidering the radius of the ſphere which paſſes through the centre of the ſmall circle circumſcribing the triangle, it is evident that this radius is perpendicular to the plane of the circle, and that it meets the ſurface of the ſphere at a point which is the pole of the ſame circle.

Therefore, putting φ for the arch which meaſures the diſtance of the circle from its pole, we have evidently $R = \text{sin.}\,\varphi$, therefore

sin.

But becauſe $p^2 = y^2 = \dfrac{\frac{m}{m-1} \times P^2 y^{n-1}}{y^{n-1} + \frac{r^{n-1}}{m-1}}$,

$y^{n-1} + \dfrac{r^{n-1}}{m-1} = \dfrac{m}{m-1} \times P^2 y^{n-3}$; $s^2 P^2 = \dfrac{r^{n-3}}{y^{n-3}}$.

But the equation for determining the apſides is

$$y^{n-1} - \frac{m}{m-1} P^2 y^{n-3} + \frac{r^{n-1}}{m-1} = 0, \text{ or}$$

$$y^{n-1} - \frac{n-1}{(n-1)s^2-2} \times s^2 P^2 y^{n-1} + \frac{2r^{n-3}}{(n-1)s^2-2} = 0;$$

whence by ſubſtitution we have

$$y^{n-1} - \frac{(n-1)r^{n-1}}{(n-1)s^2-2} + \frac{2r^{n-1}}{(n-1)s^2-2} = 0, \text{or } y^{n-1} = \frac{n-3}{(n-1)s^2-2} \times r^{n-1},$$

therefore $y = \left(\dfrac{n-3}{(n-1)s^2-2}\right)^{\frac{1}{n-1}} \times r = \left(\dfrac{n-3}{2m-2}\right)^{\frac{1}{n-1}} \times r$,

becauſe $\dfrac{2m}{n-1} = s^2$. It is evident from inſpection, that n muſt be greater than 3 and m greater than unity; the force therefore muſt vary in an higher proportion than the cube of the diſtance inverſely. If the body deſcend from the point V, then $s^2 = \dfrac{r^{n-1}}{P^2 y^{n-3}}$, muſt be greater than unity. If it aſcend, then $n - 3$ muſt be greater than $(n-1)s^2 - 2$, and therefore s^2, and conſequently s, leſs than unity.

From the above equations and obſervations, we have the following conſtruction. Fig. 158, pl. 8. Let C be the centre of force, V the point from which the body is projected in the direction VP, which makes an acute angle with CV, and with a velocity greater than would make it move in the circle VZU if projected at right angles. With the centre C and radius $Ca = \left(\dfrac{n-1}{(n-1)s^2-2}\right)^{\frac{1}{n-1}} \times r$ deſcribe the circle abd, and

$$\frac{2\sin.\frac{a}{2}\sin.\frac{b}{2}\sin.\frac{c}{2}}{3\tan.\varphi}$$

and again, by ſubſtituting for tan. φ its value already found (No. 6), we have $\frac{\pounds}{6}$ for the ſolidity of the pyramid.

8. It remains for us to conſider the area of the ſpherical triangle itſelf formed by the arcs a, b, c. There is a beautiful and known theorem, according to which, the area of a ſpherical triangle is to the whole ſurface of the ſphere, as the exceſs of the three angles of the triangle above two right angles to eight right angles. This theorem is commonly attributed to Albert Girard, who gave it in ſubſtance, in a work entitled *Invention Nouvelle en Algebre*, printed at Amſterdam in 1629; but as the proof which he gives is not rigorous, and cannot even be conſidered as an induction, the theorem ought rather to be attributed to Cavallerius, who gave it in the *Directericum generale uranometricum*, printed at Bologna in 1632, with the elegant demonſtration firſt publiſhed by Wallis, and ſince inſerted in moſt works upon trigonometry.

Let Σ denote the exceſs of the three angles of the triangle above two right angles, then, retaining the notation hitherto employed, and putting D for a right angle,

$$\Sigma = A + B + C - 2D.$$

Thus the area of the triangle of which the ſides are a, b, c and the angles A, B, C ſhall be the $\frac{\Sigma}{8D}$ part of the whole ſurface of the ſphere; and if this ſurface be conſidered as equal to 8D, we may take Σ for the area of the triangle.

9. If we ſuppoſe the ſides b and c, which contain the angle A, to be produced till each of them be a quadrant, the angles B, C will then become right angles, and the ſide a will become equal to the oppoſite angle A; then the area of the iſoſceles right angled triangle becomes A; therefore, if from it we ſubtract the former triangle, of which the ſides about the angle A are b and c, we ſhall have a ſpherical quadrilateral, the baſe of which is A, and the ſides perpendicular to that baſe $D - b$ and $D - c$, and the area of the quadrilateral ſhall be expreſſed ſimply by $B + C - 2D$.

But by the known analogies of Napier, we have in every ſpherical triangle this equation, which will be demonſtrated farther on,

tan.

Let the velocities of bodies in a circle at the diſtance r in the curve, at V and n, be $= V, v, v'$ reſpectively (fig. 152, pl. 8.); then $v' = \frac{Pv}{p}$. If therefore the ſeveral values of p, depending upon the different values of m, be ſubſtituted, we ſhall have

$$1.\ v' = \frac{v\sqrt{\left(y^{n-1} + \frac{r^{n-1}}{m-1}\right)}}{\sqrt{\left(\frac{m}{m-1}\right)} \times y^{\frac{n-1}{2}}} = v\sqrt{\frac{(m-1)y^{n-1} + r^{n-1}}{my^{\frac{n-1}{2}}}},\quad m > 1.$$

$$2.\ v' = v\sqrt{\frac{r^{n-1}}{y^{n-1}}},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ m = 1.$$

$$3.\ v' = \frac{v\sqrt{\left(\frac{r^{n-1}}{1-m} - y^{n-1}\right)}}{\sqrt{\left(\frac{m}{1-m}\right)} \times y^{\frac{n-1}{2}}} = v\sqrt{\frac{r^{n-1} - (1-m)y^{n-1}}{my^{\frac{n-1}{2}}}},\quad m < 1.$$

$$4.\ v' = \frac{V\sqrt{\left(r^{n-2} - y^{n-1}\right)}}{\sqrt{\left(\frac{n-1}{2}\right)} \times y^{\frac{n-2}{2}}},\quad m = 0, \text{ therefore } v = 0.$$

If $n = 1$, then $v' = v\sqrt{\left(1 + \frac{2}{s^2} \times \log. \frac{r}{y}\right)}$. In the 4th value of v', $m = 0 = s^2 \times \frac{n-1}{2}$, $v : V :: s : 1$; therefore $sV = v$; hence $\frac{m}{1-m} = \frac{s^2(n-1)}{2 - s^2(n-1)}$, and $\frac{v}{\sqrt{\frac{m}{1-m}}} = \frac{V}{\sqrt{\frac{n-1}{2}}}$, s being $= 0$, therefore the 4th form is evident from the 3d. Becauſe $m = s^2 \times \frac{n-1}{2}$, and $s^2 = \frac{v^2}{V^2}$, therefore $m = \frac{v^2}{V^2} \times \frac{n-1}{2}$; which value of m being ſubſtituted for it in the firſt equation gives, when reduced

$$v' = \sqrt{\left(v^2 - 2V^2 \times \frac{y^{n-1} - r^{n-1}}{(n-1)y^{n-1}}\right)} = \sqrt{\left(v^2 + 2V^2 \times \frac{r^{n-1} - y^{n-1}}{(n-1)y^{n-1}}\right)}.$$

Let g = force of gravity, and ſuppoſe the force at V, or at the diſtance r from the centre of force : force of gravity $(g) :: l : 1$,

then

If we ſubſtitute, in place of tan. $\frac{B+C}{2}$ its value found above (preceding No.), we have

$$\text{cot.}\,\frac{\Sigma}{2} = -\frac{\text{tan.}\,\frac{A}{2}\,\text{coſ.}\,\frac{b+c}{2} + \text{cot.}\,\frac{A}{2}\,\text{coſ.}\,\frac{b-c}{2}}{\text{coſ.}\,\frac{b+c}{2} - \text{coſ.}\,\frac{b-c}{2}},$$

a formula, which is eaſily transformed to this other

$$\text{cot.}\,\frac{\Sigma}{2} = \frac{\text{coſ.}\,\frac{b}{2}\,\text{coſ.}\,\frac{c}{2} + \text{ſin.}\,\frac{b}{2}\,\text{ſin.}\,\frac{c}{2}\,\text{coſ.}\,A}{\text{ſin.}\,\frac{b}{2}\,\text{ſin.}\,\frac{c}{2}\,\text{ſin.}\,A}.$$

If now we ſubſtitute in this formula the values ſin. A, and coſ. A of No. 3, we ſhall have, by dividing the numerator and denominator by ſin. $\frac{b}{2}$ ſin. $\frac{c}{2}$

$$\text{cot.}\,\frac{\Sigma}{2} = \frac{4(\text{coſ.}\,\frac{b}{2}\,\text{coſ.}\,\frac{c}{2})^2 + \text{coſ.}\,a - \text{coſ.}\,b\,\text{coſ.}\,c}{f};$$

But $2(\text{coſ.}\,\frac{b}{2})^2 = 1 + \text{coſ.}\,b$, and $2(\text{coſ.}\,\frac{c}{2})^2 = 1 + \text{coſ.}\,c$, therefore, making the neceſſary ſubſtitutions and inverting the fraction,

$$\text{tan.}\,\frac{\Sigma}{2} = \frac{f}{1 + \text{coſ.}\,a + \text{coſ.}\,b + \text{coſ.}\,c}$$

a formula of the moſt ſimple form to determine the area of a ſpherical triangle by means of its three ſides a, b, c.

11. We have ſeen (No. 7) that $\frac{f}{6}$ is the ſolidity of the triangular pyramid formed by the three radii of the ſphere which terminate in the angles of a ſpherical triangle. Let us now conſider a triangular pyramid formed by the ſame radii produced at pleaſure in ſuch a manner that they become p, q, r; and that a, b, c are the arches, or angles contained by thoſe ſtraight lines. To have the ſolidity of this pyramid, it may be conſidered as lying on one of its faces; for inſtance, that which has for its ſides the lines p, q, and to have a perpendicular P drawn from the other ſide r upon the plane of that face. It is immediately obvious, that if a is the angle contained by p and q, the area of the face which we have conſidered as the baſe of the

pyramid will be $\frac{pq\ \text{ſin.}\ a}{2}$, therefore, the ſolidity of the pyramid will be $\frac{\text{P}pq\ \text{ſin.}\ a}{6}$. But if we call θ the angle which the line r makes with the plane paſſing through p and q, it is evident that we have $\text{P} = r\ \text{ſin.}\ \theta$, therefore, the ſolidity required will be $\frac{pqr\ \text{ſin.}\ a\ \text{ſin.}\ \theta}{6}$.

The angle θ is no other than the arch let fall perpendicularly from the angle A of the ſpherical triangle upon the oppoſite ſide a; we may, therefore, expreſs ſin. θ by the ſines or coſines of a, b, c the ſides of the triangle; but for our purpoſe it is ſufficient to conſider that this value, as well as that of ſin. a, being independent of the lines p, q, r, if we make $p = 1$, $q = 1$, $r = 1$ we ſhall have the caſe of the particular pyramid we propoſe to conſider; and of which the ſolidity is $\frac{f}{6}$. Hence it follows, that we have ſin. a ſin. $\theta = f$, and therefore $\frac{pqrf}{6}$ for the ſolidity of the triangular pyramid in which the three ſides, or edges that form any one of the ſolid angles are p, q, r, and the angles contained by thoſe three ſides are A, B, C.

This expreſſion, for the ſolidity of any triangular pyramid, by means of its three ſides and contained angles, is evidently very ſimple and commodious for calculation, more eſpecially if we employ the value of f, as expreſſed by factors, in No. 4; and it may be very uſeful to determine the ſolid contents of all bodies bounded by planes, ſeeing that they may always be reſolved into triangular pyramids, as a polygon is reſolved into triangles.

12. We may add, that ſince we have found ſin. a ſin. $\theta = f$, we ſhall have ſin. $\theta = \frac{f}{\text{ſin.}\ a}$. We may thus determine by this formula the perpendicular θ in a ſpherical triangle, of which a is the baſe, and b, c the two ſides.

13. I have reſolved the preceding problems, only to ſhew the origin and uſe of ſome remarkable formulas, and more eſpecially of that function which I have denoted by f, and which deſerves in a particular manner the attention of analyſts, on account of its different applications. I now proceed to general conſiderations relating to ſpherical trigonometry, viewed analytically.

Analytical reſolutions of ſpherical triangles were at firſt only

c in the expreſſion of the radius r (No. 2), that is to ſay, in $\frac{abc}{d}$.

Let us put R for the radius of the circumſcribed circle, and h for the value which d acquires by ſubſtituting for a, b, c as above; thus we have

$$R = \frac{8\,\text{ſin.}\,\frac{a}{2} \times \text{ſin.}\,\frac{b}{2} \times \text{ſin.}\,\frac{c}{2}}{h}.$$

But we have $h^2 = 32\left(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{b}{2}\right)^2 + 32\left(\text{ſin.}\,\frac{a}{2}\,\text{ſin.}\,\frac{c}{2}\right)^2 +$

$32\left(\text{ſin.}\,\frac{b}{2}\,\text{ſin.}\,\frac{c}{2}\right)^2 - 16\left(\text{ſin.}\,\frac{a}{2}\right)^4 - 16\left(\text{sin.}\,\frac{b}{2}\right)^4 - 16\left(\text{sin.}\,\frac{c}{2}\right)^4$;

and ſubſtituting for $2\left(\text{sin.}\,\frac{a}{2}\right)^2$, $2\left(\text{sin.}\,\frac{b}{2}\right)^2$, $2\left(\text{sin.}\,\frac{c}{2}\right)^2$ their values $1 - \text{coſ.}\,a$, $1 - \text{coſ.}\,b$, $1 - \text{coſ.}\,c$ we have after proper reductions

$$\begin{aligned} h^2 = {} & 12 - 8\,\text{coſ.}\,a - 8\,\text{coſ.}\,b - 8\,\text{coſ.}\,c \\ & + 8\,\text{coſ.}\,a\,\text{coſ.}\,b + 8\,\text{coſ.}\,a\,\text{coſ.}\,c + 8\,\text{coſ.}\,b\,\text{coſ.}\,c \\ & - 4\,\text{coſ.}^2\,a - 4\,\text{coſ.}^2\,b - 4\,\text{coſ.}^2\,c \end{aligned}$$

which expreſſion may be reduced to the following

$$4f^2 + 8(1 - \text{coſ.}\,a)(1 - \text{coſ.}\,b)(1 - \text{coſ.}\,c)$$

that is to ſay, to

$$4f^2 + 64\left(\text{ſin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2}\right)^2$$

by ſubſtituting the value of f of No. 4. Thus we have

$$R = \frac{4\,\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2}}{\sqrt{\left[f^2 + 16\left(\text{sin.}\,\frac{a}{2}\,\text{sin.}\,\frac{b}{2}\,\text{sin.}\,\frac{c}{2}\right)^2\right]}}.$$

6. Now by conſidering the radius of the ſphere which paſſes through the centre of the ſmall circle circumſcribing the triangle, it is evident that this radius is perpendicular to the plane of the circle, and that it meets the ſurface of the ſphere at a point which is the pole of the ſame circle.

Therefore, putting φ for the arch which meaſures the diſtance of the circle from its pole, we have evidently $R = \text{sin.}\,\varphi$, therefore

$$4.\ \dot{t} = \pm \frac{\sqrt{\tfrac{1}{2}} \times y\dot{y}}{V\sqrt{(ry - y^2)}} = \frac{\sqrt{\tfrac{1}{2}} \times y\dot{y}}{\sqrt{(lgr)} \times \sqrt{(ry-y^2)}},\ P=0,\ m=0.$$

The fluents in all theſe caſes may be found by circular arcs and logarithms.

The correct fluent being taken for the 4th. form, gives

$$t = \frac{1}{V\sqrt{2}} \times \left\{ \begin{matrix} \sqrt{(ry-y^2)} + \text{arc whoſe verſed ſine} = r-y \\ \text{and radius} = \tfrac{1}{2} r \end{matrix} \right\}$$

$$= \frac{1}{\sqrt{(2lgr)}} \times \left\{ \begin{matrix} \sqrt{(ry-y^2)} + \text{arc whoſe verſed ſine} = r-y \\ \text{and radius} = \tfrac{1}{2} \end{matrix} \right\}.$$

and when the body is fallen to the centre, then

$$t = \frac{1}{\sqrt{(2lgr)}} \times \frac{pr}{2};\ p \text{ being} = 3{\cdot}141592.$$

Cor. 2. If $n = 3$ the values of $\dot{t}$ will become

$$1.\ \dot{t} = \pm \frac{\sqrt{\left(\frac{m}{m-1}\right)}}{v\sqrt{\left(y^2 + \frac{r^2 - mP^2}{m-1}\right)}}, \ldots\ldots\ldots\ldots\ldots m > 1.$$

$$2.\ \dot{t} = \pm \frac{y\dot{y}}{v\sqrt{(r^2 - P^2)}}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots m = 1.$$

$$3.\ \dot{t} = \pm \frac{\qquad}{v\sqrt{\left(\frac{r^2 - mP^2}{1-m} - y^2\right)}}, \ldots\ldots\ldots\ldots\ldots m < 1.$$

$$4.\ \dot{t} = - \frac{y\dot{y}}{V\sqrt{(r^2 - y^2)}} = - \frac{y\dot{y}}{\sqrt{(lgr)} \times \sqrt{(r^2-y^2)}},\ P=0,\ m=0.$$

Hence by taking the correct fluents there will ariſe

$$1.\ t = \frac{1}{v}\sqrt{\left(\frac{m}{m-1}\right)} \times \left\{ \pm \sqrt{\left(y^2 + \frac{r^2-mP^2}{m-1}\right)} \mp \sqrt{\left(\frac{m}{m-1}\right)} \times (r^2 - P^2) \right\},\ m > 1.$$

$$2.\ t = \frac{1}{2v} \times \frac{\pm y^2 \mp r^2}{\sqrt{(r^2 - P^2)}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots m = 1.$$

$$5.\ t = \frac{1}{v}\sqrt{\left(\frac{m}{1-m}\right)} \times \left\{ \mp \sqrt{\left(\frac{r^2-mP^2}{1-m} - y^2\right)} \pm \sqrt{\left[\left(\frac{m}{1-m}\right) \times (r^2 - P^2)\right]} \right\},\ m < 1.$$

$$4.\ t = \frac{1}{V} \times \sqrt{(r^2-y^2)} = \frac{r}{V} = \frac{r}{\sqrt{(lgr)}} = \frac{r^{\frac{1}{2}}}{\sqrt{(lg)}},\ \text{when } y=0,\ m=0.$$

$$\frac{2\sin.\frac{a}{2}\sin.\frac{b}{2}\sin.\frac{c}{2}}{3\tan.\varphi}$$

and again, by ſubſtituting for tan. φ its value already found (No. 6), we have $\frac{\mathcal{L}}{6}$ for the ſolidity of the pyramid.

8. It remains for us to conſider the area of the ſpherical triangle itſelf formed by the arcs a, b, c. There is a beautiful and known theorem, according to which, the area of a ſpherical triangle is to the whole ſurface of the ſphere, as the exceſs of the three angles of the triangle above two right angles to eight right angles. This theorem is commonly attributed to Albert Girard, who gave it in ſubſtance, in a work entitled *Invention Nouvelle en Algebre*, printed at Amſterdam in 1629; but as the proof which he gives is not rigorous, and cannot even be conſidered as an induction, the theorem ought rather to be attributed to Cavallerius, who gave it in the *Directericum generale uranometricum*, printed at Bologna in 1632, with the elegant demonſtration firſt publiſhed by Wallis, and ſince inſerted in moſt works upon trigonometry.

Let Σ denote the exceſs of the three angles of the triangle above two right angles, then, retaining the notation hitherto employed, and putting D for a right angle,

$$\Sigma = A + B + C - 2D.$$

Thus the area of the triangle of which the sides are a, b, c and the angles A, B, C ſhall be the $\frac{\Sigma}{8D}$ part of the whole ſurface of the ſphere; and if this ſurface be conſidered as equal to 8D, we may take Σ for the area of the triangle.

9. If we ſuppoſe the sides b and c, which contain the angle A, to be produced till each of them be a quadrant, the angles B, C will then become right angles, and the side a will become equal to the oppoſite angle A; then the area of the iſoſceles right angled triangle becomes A; therefore, if from it we ſubtract the former triangle, of which the sides about the angle A are b and c, we ſhall have a ſpherical quadrilateral, the baſe of which is A, and the sides perpendicular to that baſe $D - b$ and $D - c$, and the area of the quadrilateral ſhall be expreſſed simply by $B + C - 2D$.

But by the known analogies of Napier, we have in every ſpherical triangle this equation, which will be demonſtrated farther on,

tan.

$$\frac{mA^{n+1}}{n-1}=\frac{cm'A^{n+1}\times r^{q-n}}{(q-1)r^{q-n}},\ \text{or,}\ \frac{m}{n-1}=\frac{cm'}{q-1};$$

hence $m'=\frac{(q-1)m}{c(n-1)}$.

If therefore this value of m' be ſubſtituted in the above value of p^2, there will ariſe for the general equation of the curve

$$p^2=\frac{mP^2y^{n-1}}{r^{n-1}+c\times\frac{n-1}{q-1}\times r^{q-1}y^{n-q}+(m-1-c)\times\frac{n-1}{q-1}\times y^{n-1}}.$$

If p be ſuppoſed $=y$, then the equation for determining the diſtance of the apſides from the centre of force will become

$$\left.\begin{array}{r} y^{n-1}+\frac{c(n-1)r^{q-1}}{(m-1)(q-1)-c(n-1)}\times y^{n-q}-\frac{m(q-1)P^2}{(m-1)(q-1)-c(n-1)} \\ \times y^{n-3}+\frac{(q-1)r^{n-1}}{(m-1)(q-1)-c(n-1)} \end{array}\right\}=0.$$

The value of $\dot{z}$ will be found by an eaſy ſubſtitution.

After the ſame manner the equation of the curve, &c. &c. may be found, if three or more forces act upon a body towards the ſame centre.

Cor. If $n=2$, $q=-1$, $r=P$, and the force $\frac{B^{q+1}}{y^q}$, be ſuppoſed to act from the centre, or, which is the ſame thing, $-c$ be put for $+c$; then the equation for determining the diſtance of the apſides from the centre will become

$y^4-\frac{2+c-2m}{c}\times r^2y^2+\frac{2r^3}{c}\times y-\frac{2mr^4}{c}=0$. It is evident that one of the values of y will be $=r$, and the equation given above being divided by $y-r$, the quotient will be

$y^3+ry^2-\frac{2-2m}{c}\times r^2y+\frac{2m}{c}\times r^3=0$; by ſolving which equation, the diſtance of the other apſe from the centre may be determined.

If it be required to exterminate m, we have

$\frac{v^2}{s^2}\times\frac{1}{r}=\frac{A^{n+1}}{r^n}+\frac{B^{q+1}}{r^q}$ (Propoſition 1.) or

$$v^2=s^2\times\left(\frac{A^{n+1}}{r^{n-1}}+\frac{B^{q+1}}{r^{q-1}}\right)=\frac{2mA^{n+1}}{(n-1)r^{n-1}};$$

If we ſubſtitute, in place of tan. $\frac{B+C}{2}$ its value found above (preceding No.), we have

$$\text{cot.}\ \frac{\Sigma}{2} = -\frac{\text{tan.}\ \frac{A}{2}\ \text{coſ.}\ \frac{b+c}{2} + \text{cot.}\ \frac{A}{2}\ \text{coſ.}\ \frac{b-c}{2}}{\text{coſ.}\ \frac{b+c}{2} - \text{coſ.}\ \frac{b-c}{2}},$$

a formula, which is eaſily transformed to this other

$$\text{cot.}\ \frac{\Sigma}{2} = \frac{\text{coſ.}\ \frac{b}{2}\ \text{coſ.}\ \frac{c}{2} + \text{ſin.}\ \frac{b}{2}\ \text{ſin.}\ \frac{c}{2}\ \text{coſ.}\ A}{\text{ſin.}\ \frac{b}{2}\ \text{ſin.}\ \frac{c}{2}\ \text{ſin.}\ A}.$$

If now we ſubſtitute in this formula the values ſin. A, and coſ. A of No. 3, we ſhall have, by dividing the numerator and denominator by ſin. $\frac{b}{2}$ ſin. $\frac{c}{2}$

$$\text{cot.}\ \frac{\Sigma}{2} = \frac{4(\text{coſ.}\ \frac{b}{2}\ \text{coſ.}\ \frac{c}{2})^2 + \text{coſ.}\ a - \text{coſ.}\ b\ \text{coſ.}\ c}{f};$$

But $2(\text{coſ.}\ \frac{b}{2})^2 = 1 + \text{coſ.}\ b$, and $2(\text{coſ.}\ \frac{c}{2})^2 = 1 + \text{coſ.}\ c$, therefore, making the neceſſary ſubſtitutions and inverting the fraction,

$$\text{tan.}\ \frac{\Sigma}{2} = \frac{f}{1 + \text{coſ.}\ a + \text{coſ.}\ b + \text{coſ.}\ c}$$

a formula of the moſt ſimple form to determine the area of a ſpherical triangle by means of its three ſides a, b, c.

11. We have ſeen (No. 7) that $\frac{f}{6}$ is the ſolidity of the triangular pyramid formed by the three radii of the ſphere which terminate in the angles of a ſpherical triangle. Let us now conſider a triangular pyramid formed by the ſame radii produced at pleaſure in ſuch a manner that they become p, q, r; and that a, b, c are the arches, or angles contained by thoſe ſtraight lines. To have the ſolidity of this pyramid, it may be conſidered as lying on one of its faces; for inſtance, that which has for its ſides the lines p, q, and to have a perpendicular P drawn from the other ſide r upon the plane of that face. It is immediately obvious, that if a is the angle contained by p and q, the area of the face which we have conſidered as the baſe of the

pyramid will be $\frac{pq \text{ ſin. } a}{2}$, therefore, the ſolidity of the pyramid will be $\frac{Ppq \text{ ſin. } a}{6}$. But if we call θ the angle which the line r makes with the plane paſſing through p and q, it is evident that we have $P = r$ ſin. θ, therefore, the ſolidity required will be $\frac{pqr \text{ ſin. } a \text{ ſin. } \theta}{6}$.

The angle θ is no other than the arch let fall perpendicularly from the angle A of the ſpherical triangle upon the oppoſite ſide a; we may, therefore, expreſs ſin. θ by the ſines or coſines of a, b, c the ſides of the triangle; but for our purpoſe it is ſufficient to conſider that this value, as well as that of ſin. a, being independent of the lines p, q, r, if we make $p = 1, q = 1$, $r = 1$ we ſhall have the caſe of the particular pyramid we propoſe to conſider; and of which the ſolidity is $\frac{f}{6}$. Hence it follows, that we have ſin. a ſin. $\theta = f$, and therefore $\frac{pqrf}{6}$ for the ſolidity of the triangular pyramid in which the three ſides, or edges that form any one of the ſolid angles are p, q, r, and the angles contained by thoſe three ſides are A, B, C.

This expreſſion, for the ſolidity of any triangular pyramid, by means of its three ſides and contained angles, is evidently very ſimple and commodious for calculation, more eſpecially if we employ the value of f, as expreſſed by factors, in No. 4; and it may be very uſeful to determine the ſolid contents of all bodies bounded by planes, ſeeing that they may always be reſolved into triangular pyramids, as a polygon is reſolved into triangles.

12. We may add, that ſince we have found ſin. a ſin. $\theta = f$, we ſhall have ſin. $\theta = \frac{f}{\text{ſin. } a}$. We may thus determine by this formula the perpendicular θ in a ſpherical triangle, of which a is the baſe, and b, c the two ſides.

13. I have reſolved the preceding problems, only to ſhew the origin and uſe of ſome remarkable formulas, and more eſpecially of that function which I have denoted by f, and which deſerves in a particular manner the attention of analyſts, on account of its different applications. I now proceed to general conſiderations relating to ſpherical trigonometry, viewed analytically.

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$$\frac{2\sin.\frac{a}{2}\sin.\frac{b}{2}\sin.\frac{c}{2}}{3\tan.\varphi}$$

and again, by ſubſtituting for tan. φ its value already found (No. 6), we have $\frac{\mathcal{L}}{6}$ for the ſolidity of the pyramid.

8. It remains for us to conſider the area of the ſpherical triangle itſelf formed by the arcs *a*, *b*, *c*. There is a beautiful and known theorem, according to which, the area of a ſpherical triangle is to the whole ſurface of the ſphere, as the exceſs of the three angles of the triangle above two right angles to eight right angles. This theorem is commonly attributed to Albert Girard, who gave it in ſubſtance, in a work entitled *Invention Nouvelle en Algebre*, printed at Amſterdam in 1629; but as the proof which he gives is not rigorous, and cannot even be conſidered as an induction, the theorem ought rather to be attributed to Cavallerius, who gave it in the *Directericum generale uranometricum*, printed at Bologna in 1632, with the elegant demonſtration firſt publiſhed by Wallis, and ſince inſerted in moſt works upon trigonometry.

Let Σ denote the exceſs of the three angles of the triangle above two right angles, then, retaining the notation hitherto employed, and putting D for a right angle,

$$\Sigma = A + B + C - 2D.$$

Thus the area of the triangle of which the ſides are *a*, *b*, *c* and the angles A, B, C ſhall be the $\frac{\Sigma}{8D}$ part of the whole ſurface of the ſphere; and if this ſurface be conſidered as equal to 8D, we may take Σ for the area of the triangle.

9. If we ſuppoſe the ſides *b* and *c*, which contain the angle A, to be produced till each of them be a quadrant, the angles B, C will then become right angles, and the ſide *a* will become equal to the oppoſite angle A; then the area of the iſoſceles right angled triangle becomes A; therefore, if from it we ſubtract the former triangle, of which the ſides about the angle A are *b* and *c*, we ſhall have a ſpherical quadrilateral, the baſe of which is A, and the ſides perpendicular to that baſe D — *b* and D — *c*, and the area of the quadrilateral ſhall be expreſſed ſimply by B + C — 2D.

But by the known analogies of Napier, we have in every ſpherical triangle this equation, which will be demonſtrated farther on,

$$\tan.\frac{B+C}{2}=\frac{\text{coſ.}\frac{b-c}{2}}{\text{coſ.}\frac{b+c}{2}}\cot.\frac{A}{2}.$$

Therefore, if σ denote the ſurface of the quadrilateral we are conſidering, we ſhall have

$$\tan.\frac{\sigma}{2}=\cot.\frac{B+C}{2}=\frac{1}{\tan.\frac{B+C}{2}};\text{ therefore,}$$

$$\tan.\frac{\sigma}{2}=\frac{\text{coſ.}\frac{b+c}{2}}{\text{coſ.}\frac{b-c}{2}}\tan.\frac{A}{2};$$

and if we put β and γ for the two ſides of the quadrilateral perpendicular to the baſe A, ſo that $\beta=D-b$ and $\gamma=D-c$ we have in order to determine the area σ this formula

$$\tan.\frac{\sigma}{2}=\frac{\sin.\frac{\beta+\gamma}{2}}{\text{coſ.}\frac{\beta-\gamma}{2}}\tan.\frac{A}{2}.$$

This formula correſponds to the known formula $\sigma=\frac{\beta+\gamma}{2}A$ for a rectilineal quadrilateral of which A is the baſe, β and γ the two vertical ſides, and σ the area; and as the latter is of the greateſt utility in the menſuration of plane ſurfaces bounded by ſtraight lines, ſo the former is equally uſeful in the menſuration of ſpherical ſurfaces bounded by arches of great circles. Thus it may be employed with much advantage to determine the extent of a country, when the latitudes and longitudes of ſeveral points in its circumference are known; for by connecting thoſe points by arches of great circles, we have a ſpherical polygon, the area of which may be eaſily found, by reſolving it into quadrilaterals formed by circles of latitudes and arches of the equator intercepted between thoſe circles.

10. But if it was required to expreſs the area Σ by the three ſides a, b, c of the ſpherical triangle, it is only neceſſary to conſider, that ſince $\Sigma=A+B+C-2D$ we have

$$\cot.\frac{\Sigma}{2}=-\tan.\frac{A+B+C}{2}$$

$$=-\frac{\tan.\frac{A}{2}+\tan.\frac{B+C}{2}}{1-\tan.\frac{A}{2}\tan.\frac{B+C}{2}}.$$

If we ſubſtitute, in place of tan. $\frac{B+C}{2}$ its value found above (preceding No.), we have

$$\text{cot.}\ \frac{\Sigma}{2} = -\frac{\text{tan.}\ \frac{A}{2}\,\text{coſ.}\ \frac{b+c}{2} + \text{cot.}\ \frac{A}{2}\,\text{coſ.}\ \frac{b-c}{2}}{\text{coſ.}\ \frac{b+c}{2} - \text{coſ.}\ \frac{b-c}{2}},$$

a formula, which is eaſily transformed to this other

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The curve therefore is a parabola, whoſe axis is perpendicular to the plane, latus rectum $= \frac{b^2}{\frac{1}{2}A}$ and force $=$ A, meaſured by the velocity generated in the time (1).

PROPOSITION V.

If a body revolve round a centre, and be acted upon by a force tending to that centre which varies as the *n*th power of the diſtance inverſely, and whoſe quantity at an apſe, the diſtance of which apſe being r, is $= 1$; likewiſe, if another body, beſides being ſubject to this force, be acted upon by an additional one, which varies inverſely as the *q*th power of the diſtance, its value being c at the ſame diſtance r; it is required to inveſtigate a general expreſſion for the ratio of the angular velocities of the two bodies when at the ſame diſtance from the centre of force.

Let the required ratio be that of F : G. Put $y =$ common diſtance, $p =$ perpendicular upon the tangent to the orbit deſcribed by the ſingle force, and $p' =$ perpendicular upon the tangent to the other orbit, drawn from the common centre.

It is evident by propoſition 1, that

$$F^2 : G^2 :: \frac{p^2\dot{y}^2}{(y^2-p^2)y^2} : \frac{p'^2\dot{y}^2}{(y^2-p'^2)y^2} :: \frac{p^2}{y^2-p^2} : \frac{p'^2}{y^2-p'^2}.$$

But (by prop. 1*ſt*. and 3*d*.) $p^2 = \dfrac{\frac{m}{1-m} \times r^2y^{n-1}}{\frac{r^{n-1}}{1-m} - y^{n-1}}$, and

$$p'^2 = \frac{Mr^2y^{n-1}}{r^{n-1} + c \times \frac{n-1}{q-1} \times r^{q-1}y^{n-q} + \left(M - 1 - c \times \frac{n-1}{q-1}\right) \times y^{n-1}},$$

writing M here inſtead of m in the third propoſition.

Hence, by ſubſtitution and reduction, $F^2 : G^2$

$$:: \frac{m}{(m-1)y^2 - mr^2 + r^{r-1}y^{3-n}}$$

$$: \frac{M}{\left(M-1-c \times \frac{n-1}{q-1}\right) \times y^2 + c \times \frac{n-1}{q-1} \times r^{q-1}y^{3-q} + r^{n-1}y^{3-n} - mr^2}.$$

But $m = \frac{n-1}{2} \times s^2$ and $M = \frac{(n-1)(1+c)}{2} \times S^2$ (writing S, for s in prop. 3.) therefore $F^2 : G^2$

$$:: \frac{s^2}{\left\{(n-1)s^2 - 2\right\} \times y^2 - (n-1)\, s^2 r^2 + 2r^{n-1} y^{3-n}}$$

$$: \frac{(1+c)\, S^2}{\left\{(n-1)(1+c)S^2 - 2 - 2c . \frac{n-1}{q-1}\right\} . y^2 + 2c . \frac{n-1}{q-1} r^{q-1} y^{3-q} + 2r^{n-1} y^{3-n} - (n-1)(1+c)S^2 r^2}$$

which is general, whatever be the values of n and q, s and S.

Cor. 1. If $q = 3$, then $F^2 : G^2$

$$:: \frac{s^2}{\left\{(n-1)\, s^2 - 2\right\} \times y^2 - (n-1) s^2 r^2 + 2r^{n-1} y^{3-n}}$$

$$: \frac{(1+c)\, S^2}{\left\{(n-1)\,(1+c)S^2 - 2 - c(n-1)\right\} . y^2 - \left\{(n-1)(1+c)S^2 - c(n-1)\right\} . r^2 + 2r^{n-1} y^{3-n}}$$

where it is evident that if we make $(n-1)\,(1+c)\, S^2 - c\,(n-1) = (n-1)s^2$, then the denominators will be equal, and therefore $F^2 : G^2 :: s^2 : (1+c)\, S^2$, which is a conſtant ratio; and therefore when $q = 3$, or when the additional force varies as the cube of the diſtance reciprocally, the angles, which lines drawn from the bodies to the centre of force, make with the line paſſing through the apſe, are in a conſtant ratio to each other, whatever be their common diſtance (the condition mentioned being obſerved).

Cor. 2. Becauſe $(n-1)\,(1+c)\, S^2 - c\,(n-1) = (n-1)s^2$, therefore

$S^2 = \frac{s^2 + c}{1+c}$. Let R = radius of curvature and v = velocity at the apſe in the orbit deſcribed by one force; then the centripetal force at the diſtance r, $= \frac{v^2}{R} = \frac{v^2}{s^2 r}$, therefore

$s^2 = \frac{R}{r}$ and $S^2 = \frac{s^2 + c}{1+c} = \frac{R + rc}{r^2\,(1+c)}$; hence

$F^2 : G^2 :: \frac{R}{r} : \frac{R + rc}{r} :: R : R + rc$, or

$F^2 : \frac{G^2 - F^2}{r} :: R : c = \frac{G^2 - F^2}{rF^2} \times R$ = additional force at the apſe. But becauſe $q = 3$,

$y^3 : r^3 :: \frac{G^2 - F^2}{rF^2} \times R : \frac{G^2 - F^2}{F^2} \times \frac{Rr^2}{y^3}$ = additional force at the diſtance y, and the force in the firſt orbit at the ſame diſtance $= \frac{r^n}{y^n}$, therefore

the whole compound force $= \frac{r^n}{y^n} + \frac{G^2 - F^2}{F^2} \times \frac{Rr^2}{y^3}$.

COR. 3. If inſtead of (1), $\frac{v^2}{R}$ be ſuppoſed = force in the firſt orbit at the diſtance r, it will eaſily appear that the compound force in the other orbit, at the diſtance y, will be $= \frac{r^n v^2}{Ry^n} + \frac{G^2 - F^2}{F^2} \times \frac{v^2 r^2}{y^3}$; the ſame expreſſion that is found after a very different manner, Principia, Book 1*ſt*. Sect. 9*th*.

* * *

NOTE. (*Referred to in Page 45*).

Let VABD (fig 159, pl. 8.) be a trajectory deſcribed by a body round a centre of force C, V an apſe, and A the next following one; V being at a greater diſtance from the centre than A. It is evident that if the body were projected from A, at right angles to CA, and with the ſame velocity it had when it arrived there, it would accurately deſcribe the arch AV, and have the ſame velocity at V that it firſt began with. For during the time of moving over any particle of the curve, the force acting upon the body, and the direction of the force are the ſame in both caſes; hence the concluſion is clear. But if the body, inſtead of moving towards V, be projected in a contrary direction, at right angles to AC, and with the ſame velocity, an arch AB, equal and ſimilar to AV, will be deſcribed; B being an apſe, and CB = CV. Hence in the above equation y can have but two different values; but as theſe *may* be in oppoſite directions, two may be poſitive and two equal to them and negative. The other roots, if any, muſt either be impoſſible, or relate to ſuch parts of the algebraical curve as have their concavity turned from the centre of force, being deſcribed by a centrifugal force, or ſuch parts as are ſeparated from that part in which the body moves; that is, it cannot be a curve of continued curvature, as that *muſt be* in which the body moves.

The ſame concluſion may be deduced immediately from the nature of the equation found above for determining the apſides. For making the equation of the limits = 0, we have $y = \pm P \times \sqrt{\left(\frac{m}{m-1} \times \frac{n-3}{n-1}\right)}$. Whence it is evident, that there can be no more than four roots, two poſitive and two negative. *But to diſcover in every* caſe the number of poſſible roots is a problem of conſiderable difficulty. **Dr. Waring** has pointed out the method of doing this (*Meditationes Algebraicæ, Prob.* 14.); but as his manner of writing is, in general, very conciſe, an eaſy inveſtigation of the ſeveral concluſions there deduced may not be improper in this place.

It is well known, that if varying the coefficients of an equation two roots become equal, the next inſtant they will be impoſſible, and immediately before becoming equal they will be real and unequal.

This being granted, let it be propoſed to find the number of poſſible roots in the equation $x^n - Ax^m - B = 0$, which is an equation of the ſame kind with that for determining the apſides. Firſt, find the equation of limits, and make it $= 0$, viz.

$$nx^{n-1} - mAx^{m-1} = 0, \text{ then } x = \left(\frac{m}{n}\right)^{\frac{1}{n-m}} \times A^{\frac{1}{n-m}};$$

hence x has two values, if n and m be both even numbers, and $-A$ a negative number, wherefore the number of real roots in the given equation cannot exceed four.

Multiply the firſt equation by n, and the ſecond by x, and take the difference of the products, then $(m-n) \times Ax^m - nB = 0$; from whence $x = \left(\frac{n}{m-n}\right)^{\frac{1}{m}} \times \left(\frac{B}{A}\right)^{\frac{1}{m}}$, which will give two other limits, if $-B$ be poſitive, and the reſt as above.

If the given equation have two equal roots, they will coincide both with the firſt and ſecond values of x juſt found; therefore theſe values muſt be equal to each other; and converſely, if theſe values be equal, the given equation muſt have two equal roots; and if the condition abovementioned take place, *viz.* that n and m, be both even numbers, $-A$ negative, and $-B$ poſitive, then there will be two pairs of equal roots. Make them equal to each other, then by reduction it will eaſily appear that

$$\frac{(m-n)^{n-m}}{(n)^{n-m}} \times A^n - \frac{(n)^m}{(m)^m} \times B^{n-m} = 0;$$

hence, by making this expreſſion poſitive, or negative, according to circumſtances, the number of poſſible roots may be obtained.

1. If n be an even number, and $-B$ negative, it is evident that the equation has two real roots. If n be even and m odd, and $-B$ affirmative, and at the ſame time

$$\frac{(m-n)^{n-m}}{(n)^{n-m}} \times A^n - \frac{(n)^m}{(m)^m} \times B^{n-m}$$

be negative, then there will be two poſſible roots; otherwiſe none. For the firſt part of the above expreſſion is in this caſe negative, and the other part poſitive; but the firſt part muſt be greater than the ſecond, if the roots be real; becauſe if it vaniſh, all the roots are impoſſible; hence the concluſion is clear.

Let n and m, be even numbers, and $-A$, and $-B$ poſitive quantities; it is evident that the equation can have no real roots; for in this caſe no quantity ſubſtituted for x can make the reſult $= 0$. But if $-B$ be affirmative, and $-A$ negative, and at the ſame time

$$\frac{(m-n)^{n-m}}{(n)^{n-m}} \times A^n - \frac{(n)^m}{(m)^m} \times B^{n-m}$$

be affirmative then there will be

four

our real roots, otherwiſe none. For, from what has been ſaid above, there will be two pairs of equal roots when the above expreſſion $= 0$, wherefore by making it affirmative there will be four real roots. That the firſt part of the expreſſion muſt be greater than the ſecond will be evident from obſerving, that if $A = 0$, then all the roots will be impoſſible.

2 If both n and m be odd numbers, and

$$\frac{(m-n)^{n-m}}{(n)^{n-m}} \times A^{n} - \frac{(n)^{m}}{(m)^{m}} \times B^{n-m}$$ affirmative, then there will be three

poſſible roots; otherwiſe only one, Becauſe n is an odd number there muſt be one real root: and if the above expreſſion $= 0$, there will be other two roots equal. But if A and B be taken ſuch as to make it affirmative, there will be other two real roots. For $n - m$ is an even number, therefore the firſt part of the expreſſion is poſitive; and if $A = 0$, there will be only one real root, and the above expreſſion would in that caſe be negative. Therefore, &c.

If n be odd, and m even, and A and B, have the ſame ſign, then the equation will have only one real root; this requires no proof. But if they have different ſigns, and at the ſame time

$$\frac{(m-n)^{n-m}}{(n)^{n-m}} \times A^{n} - \frac{(n)^{m}}{(m)^{m}} \times B^{n-m}$$ have a contrary ſign to B, then the

equation will have three real roots; otherwiſe but one. If the equation have this form, *viz.*

$x^{n} + Ax^{m} - B = 0$, then $\frac{(m-n)^{n-m}}{(n)^{n-m}} \times A^{n}$ will be poſitive, and it muſt

be greater than $\frac{(n)^{m}}{(m)^{m}} \times B^{n-m}$, for reaſons given above; therefore the whole

is poſitive.

But if it be of this form, *viz.* $x^{n} - Ax^{m} + B = 0$, then the firſt part of the general expreſſion is negative, and the ſecond part poſitive, wherefore the reſult muſt be negative.

N.B. All the figures referred to in this Article are to be found in plate VIII. (A) and not in plate VIII.

END OF VOL. I.

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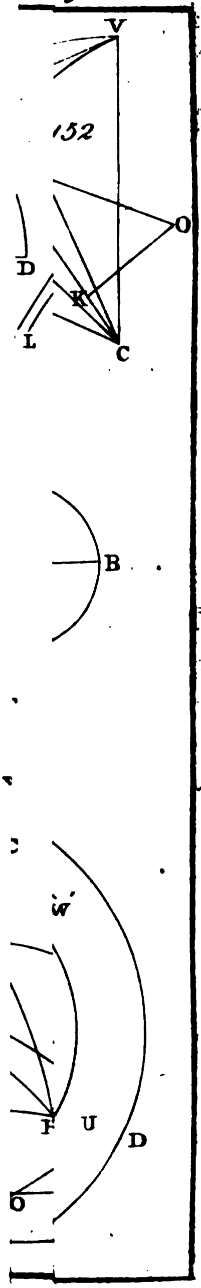
Fig. 152 to 162.



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